

# Inference on Functionals under First Order Degeneracy

Qihui Chen

School of Management and Economics  
The Chinese University of Hong Kong, Shenzhen  
qihuichen@cuhk.edu.cn

Zheng Fang\*

Department of Economics  
Texas A&M University  
zfang@tamu.edu

January 9, 2018

## Abstract

This paper presents a unified second order asymptotic framework for conducting inference on parameters of the form  $\phi(\theta_0)$ , where  $\theta_0$  is unknown but can be estimated by  $\hat{\theta}_n$ , and  $\phi$  is a known map that admits null first order derivative at  $\theta_0$ . For a large number of examples in the literature, the second order Delta method reveals a nondegenerate weak limit for the plug-in estimator  $\phi(\hat{\theta}_n)$ . We show, however, that the “standard” bootstrap is consistent if and only if the second order derivative  $\phi''_{\theta_0} = 0$  under regularity conditions, i.e., the standard bootstrap is inconsistent if  $\phi''_{\theta_0} \neq 0$ , and provides degenerate limits unhelpful for inference otherwise. We thus identify a source of bootstrap failures distinct from that in Fang and Santos (2016) because the problem (of consistently bootstrapping a *nondegenerate* limit) persists even if  $\phi$  is differentiable. We show that the correction procedure in Babu (1984) can be extended to our general setup. Alternatively, a modified bootstrap is proposed when the map is *in addition* second order nondifferentiable. Both are shown to provide local size control under some conditions. As an illustration, we develop a test of common conditional heteroskedastic (CH) features, a setting with both degeneracy and nondifferentiability – the latter is because the Jacobian matrix is degenerate at zero and we allow the existence of multiple common CH features.

KEYWORDS: First order degeneracy, Second order Delta method, Bootstrap consistency, Babu correction, Common CH features,  $J$ -test.

---

\*We would like to thank Brendan Beare, Andres Santos, Yixiao Sun and three anonymous referees for valuable suggestions that have helped greatly improve this paper. We are also grateful for Xiaohong Chen, Qi Li and seminar participants for helpful discussions and comments.