Online Appendix

Dynamic Pricing with Loss Averse Consumers and Peak-End Anchoring

Javad Nasiry
Hong Kong University of Science and Technology
Information Systems, Business Statistics and Operations Management Department. nasiry@ust.hk

Ioana Popescu
INSEAD, Decision Sciences Area. ioana.popescu@insead.edu

Appendix: Proofs

Proof of Lemma 1: For \( p_l \leq p^h \) and \( r^l \leq r^h \), we need to show that:

\[
\pi(p^h, r^h) - \pi(p^l, r^h) \geq \pi(p^h, r^l) - \pi(p^l, r^l).
\] (10)

We consider all possible cases: (1) \( p^l \leq p^h \leq r^l \leq r^h \), (2) \( p^l \leq r^l \leq p^h \leq r^h \), (3) \( p^l \leq r^l \leq r^h \leq p^h \), (4) \( r^l \leq p^l \leq p^h \leq r^h \), (5) \( r^l \leq r^h \leq p^l \leq p^h \), (6) \( r^l \leq r^h \leq p^l \leq p^h \). Cases 1 and 6 follow because \( \pi_\gamma \) or \( \pi_\lambda \), are supermodular in \((p, r)\). After rearranging terms, (10) simplifies in each case as follows:

Case 2: \( \gamma(r^h - p^h)p^h + \lambda(p^h - r^l)p^h \geq \gamma(r^h - r^l)p^l \). This holds because \( \lambda > \gamma \) and \( p^h \geq r^l \).

Case 3: \( -\lambda(p^h - r^h)p^h - \gamma(r^h - r^l)p^l \geq -\lambda(p^h - r^h)p^h - \gamma(r^l - p^l)p^l \), or, \( \lambda p^h (r^h - r^l) \geq \gamma p^l (r^h - r^l) \).

Case 4: \( p^h \left( \gamma(r^h - p^h) + \lambda(p^h - r^l) \right) \geq p^l \left( \lambda(p^l - r^l) + \gamma(r^h - p^l) \right) \). It is enough to show that \( \gamma(r^h - p^h) + \lambda(p^h - r^l) \geq \lambda(p^l - r^l) + \gamma(r^h - p^l) \), which holds because \( \lambda(p^h - p^l) \geq \gamma(p^h - p^l) \).

Case 5: \( \lambda p^h (r^h - r^l) \geq \lambda p^l (p^l - r^l) + \gamma(r^h - p^l)p^l \). Because \( \lambda \geq \gamma \) and \( r^h \geq p^l \), this is implied by \( p^h (r^h - r^l) \geq p^l (p^l - r^l) + (r^h - p^l)p^l = p^l (r^l - r^l) \).

Proof of Lemma 2: (i) For all \( p \geq m \), the function \( J_m^\nu(p) \) uniquely solves \( T J_m^\nu(p) = J_m^\nu(p) \), where the operator \( T \) is defined for any continuous function \( f \) over \([m, \overline{p}]\) by:

\[
Tf(p_{l-1}) = \max_{p_l \in \mathbb{P}} \left\{ (1-\nu)\pi_\lambda(p_l, r_l) + \nu \pi_\gamma(p_l, p_{l-1}) + \beta f(p_l) \right\}.
\] (11)

Moreover, \( \lim_{n \to \infty} T^n f(p) = J_m^\nu(p) \); see Stokey and Lucas (1989, Thm 4.6).
We argue below that, for all $p \geq m$, $J(m,p) \leq T J(m,p)$. This further implies $J(m,p) \leq T^n J(m,p) \leq \lim_{n \to \infty} T^n J(m,p) = J^{**}_m(p)$, concluding the proof. Indeed, for $m_{t-1} \leq p_{t-1}$, we have:

$$J(m_{t-1}, p_{t-1}) = \max_{p \in \mathcal{P}} \left\{ \pi(p_t, r_t) + \beta J(m_{t-1}, p_t) \right\}$$

$$\leq \max_{p \in \mathcal{P}} \left\{ (1 - \nu)\pi_{\lambda}(p_t, r_t) + \nu \pi_{\gamma}(p_t, r_t) + \beta J(m_{t-1}, p_t) \right\}$$

$$\leq \max_{p \in \mathcal{P}} \left\{ (1 - \nu)\pi_{\lambda}(p_t, r_t) + \nu \pi_{\gamma}(p_t, p_{t-1}) + \beta J(m_{t-1}, p_t) \right\} = T J(m_{t-1}, p_{t-1}).$$

The first inequality above holds because the value function is increasing and $\pi = \min(\pi_{\lambda}, \pi_{\gamma}) \leq (1 - \nu)\pi_{\lambda} + \nu \pi_{\gamma}$. The second inequality holds because $p_{t-1} \geq r_t = \theta m_{t-1} + (1 - \theta)p_{t-1}$, for $p_{t-1} \geq m_{t-1}$. Finally, the last equality follows by the definition of $T$ in (11) applied to $f(p) = J(m,p)$.

(ii) It is easy to check that, if Problem (7) admits an interior steady state, this solves the Euler Equation (8). Moreover, this equation admits a unique solution, because $\pi'_0(p) < 0$ and the coefficient of $p$ in (8) can be written as $-(1 - \beta)((1 - \nu)\lambda(1 + \theta) + \nu \gamma) \leq 0$. It remains to verify that a steady state exists, and must be interior. Existence of a steady state follows from supermodularity of the objective function, because $\pi_{\lambda}$ and $\pi_{\gamma}$ are supermodular in $(p, r)$. By Topkis Theorem (Topkis 1998, Thm. 2.8.2), this implies that the pricing paths of Problem (7) are monotonic on the bounded domain $\mathcal{P}$, hence converge to a steady state price $p^{**}$.

We further argue that a steady state must be interior. First, $p^{**} = 0$ cannot be a steady state because any non-zero pricing strategy achieves positive profits. Second, $p^{**} < \bar{p}$ for any steady state of Problem (7). This is because $\pi_0(p)$ is non-monotone, hence, its largest maximizer $\hat{p}$ is interior, i.e. $\hat{p} < \bar{p}$. Moreover, concavity of $\pi_0$ implies: $J''(\hat{p}) \geq \frac{\pi(\hat{p})}{1 - \beta} \geq \frac{\pi(p^{**})}{1 - \beta} = J''(p^{**})$. Finally, because $J''$ is increasing, we conclude that $p^{**} \leq \hat{p} < \bar{p}$, so $p^{**}$ is interior and solves the Euler Equation (8).

Finally, by definition, $p^{**}_\lambda(m)$ solves (8) for $\nu = 0$. This has a unique solution because the LHS is strictly decreasing in $p$, positive at $p = 0$ and negative at $p = \bar{p}$.

(iii) Substituting $p = m$ in (8), we have $L(m, \nu) = \pi'_0(m) - \lambda[(1 - \nu)(1 - \beta(1 - \theta)) + \nu(1 - \beta)]m = 0$; (4) and (5) translate to $L(m, 0) = 0$ and $L(m, 1) = 0$. Because $L(m, \nu)$ is decreasing in $m$, for all $m \in [\underline{m}, \overline{m}]$, $L(m, 0) \leq 0$ and $L(m, 1) \geq 0$. The result follows because $L(m, \nu)$ is continuous in $\nu$.

**Proof of Lemma 3:** (i) We first show that $p^{**}_\lambda(m)$, as defined by (6), is feasible, i.e. $p^{**}_\lambda(m) \geq m$ for $m \in [0, \overline{m}]$. Note that $p^{**}_\lambda(m)$ is increasing in $m$ and single crosses the identity line from above at $\overline{m}$, defined by (4). Feasibility follows because, at $m = 0$, (6) has a unique positive solution, $p^{**}_\lambda(0)$. 

2
For $m \in [0, m]$, the constant pricing policy $p_t \equiv p^*_\lambda(m)$ is optimal for Problem (7) with $\nu = 0$, and feasible for Problem (3). Because $m \leq m$, $\min(m, p^*_\lambda(m)) = m$, and $r = \theta m + (1 - \theta)p^*_\lambda(m) \leq p^*_\lambda(m)$, which implies $\pi = \min(\pi_\lambda, \pi_\gamma) = \pi_\lambda$. This constant pricing policy yields the same value in both problems, so it is also optimal for Problem (3), and $(m, p^*_\lambda(m))$ is a steady state of (3).

(ii) For $m \in [m, m]$, the policy $p_t \equiv m$ is optimal for Problem (7), feasible for (3) ($\pi_\lambda = \pi_\gamma$ along this path), and yields the same value in both problems. Therefore $(m, m)$ is a steady state of (3).

(iii) For any steady state $(m, p)$, either $m = p$, or $m < p$. In the second case, $r < p$ and starting at $(m, p)$, the price path gives a consistent perception of loss. Therefore, the steady state price must be the same as that of Problem (7), with $\nu = 0$, i.e. $p = p^*_\lambda(m)$. Thus Problem (3) has only two types of steady states. It remains to identify the regions where each is relevant.

First assume $m < m$. We show by contradiction that $(m, m)$ cannot be a steady state of Problem (3). If $(m, m)$ is a steady state, the profit from charging $p_t \equiv m$ exceeds that of $p_t = m + \delta$:

$$\frac{\pi_0(m)}{1 - \beta} \geq \pi_0(m + \delta) - \lambda \delta(m + \delta) + \frac{\beta}{1 - \beta}(\pi_0(m + \delta) - \lambda(m + \delta - r)(m + \delta)),$$

where $r = \theta m + (1 - \theta)(m + \delta)$. This reduces to: $\pi_0(m + \delta) - \pi_0(m) \leq \lambda \delta(m + \delta)(1 - \beta(1 - \theta))$. Dividing both sides by $\delta$ and letting $\delta \to 0$ implies $\pi_0'(m) \leq \lambda(1 - \beta(1 - \theta))m$, with equality for $m = m$ (see (4)). Because the LHS is strictly decreasing in $m$, it follows that $m \geq m$, a contradiction. We conclude that, for $m < m$, the only possible steady state for Problem (3) is $(m, p^*_\lambda(m))$.

Moreover, because $p^*_\lambda(m) < m$ for $m > m$, it follows that, for $m \geq m$, the only possible steady state is $(m, m)$. We prove by contradiction that $(m, m)$ cannot be a steady state for $m > m$. If $(m, m)$ is a steady state, the profit from charging a constant price $p_t \equiv m$ exceeds the profit along the alternative path $p_t = m - \delta$, for all $t$, i.e.: $\frac{\pi_0(m)}{1 - \beta} \geq \pi_0(m - \delta) + \gamma \delta(m - \delta) + \frac{\beta}{1 - \beta}\pi_0(m - \delta)$, or $\pi_0(m) - \pi_0(m - \delta) \geq \gamma \delta(1 - \beta)(m - \delta)$. Dividing by $\delta$ and letting $\delta \to 0$ gives $\pi_0'(m) \geq \gamma(1 - \beta)m$, with equality for $m = m$ (see (5)). Strict concavity of $\pi_0$ implies $m \leq m$, a contradiction. We conclude that steady states of the form $(m, m)$ can only be relevant when $m \leq m \leq m$.

**Proof of Lemma 4(i)**: We prove this in two parts, depending if $m_0 \in R_1$, or $m_0 \in R_2$. For $m_0 \in R_1$, we consider two cases, depending if $p_0 \geq p^*_\lambda(m_0)$, which leads us to define the regions $R_{1a} = \{(m, p) \mid p \geq p^*_\lambda(m), m \leq m\}$ and $R_{1b} = \{(m, p) \mid p \leq p^*_\lambda(m), m \leq m, p\}$ (see
This contradicts the fact that $p_t^{\ast}(m_0)$ converges monotonically to the steady state price, $p_{\lambda}^{\ast}(m_0)$. Because $p_0 < p_{\lambda}^{\ast}(m_0)$, the optimal price path for $J'=0$ increases to this steady state, and $p_t^{\ast}(r_t) \leq p_{\lambda}^{\ast}(m_0)$ for all $t$. This implies that $m_t = m_0$ along this path, and thus $r_t = \theta m_0 + (1 - \theta) p_{t-1} \leq p_{t-1} \leq p_t$. Therefore, $\pi = \min(\pi_{\lambda}, \pi_{\gamma}) = \pi_{\lambda}$, and this path is feasible for (3), and yields the same value; so the same path is also optimal for Problem (3). This result is stronger than stated in the claim, because it guarantees also the existence of the steady state, and the monotonicity of the price path. \[ \square \]

Claim 2. Given $(m_0, p_0) \in \overline{R}_{1b}$, then $p_t \geq p_{\lambda}^{\ast}(m_0) \geq m_0$ for any $t$.

Proof. For $m_0 \leq m$, we show that if at any time it is optimal to price below $p_{\lambda}^{\ast}(m_0)$, then $(m_0, p_{\lambda}^{\ast}(m_0))$ cannot be a steady state of Problem (3) which is a contradiction.

Let $r_t^* = \theta m_0 + (1 - \theta) p_{\lambda}^{\ast}(m_0)$. We show that $p_t = p^*(m_0, p_0) \geq p_{\lambda}^{\ast}(m_0)$, and conclude by induction that $p_t \geq p_{\lambda}^{\ast}(m_0)$. Assume by contradiction, $p_t < p_{\lambda}^{\ast}(m_0)$. Then:

$$\pi(p_1, r_1) - \pi(p_{\lambda}^{\ast}(m_0), r_1) > \beta \Delta J = J(m_0, p_{\lambda}^{\ast}(m_0)) - J(\min(p_1, m_0), p_1), \quad (12)$$

Because $p_0 > p_{\lambda}^{\ast}(m_0) > p_t$, we have $r_1 > r_t^*$. Supermodularity of $\pi(p, r)$ (Lemma 1), then implies:

$$\pi(p_1, r_1^*) - \pi(p_{\lambda}^{\ast}(m_0), r_1^*) \geq \pi(p_1, r_1) - \pi(p_{\lambda}^{\ast}(m_0), r_1). \quad (13)$$

Because $p_{\lambda}^{\ast}(m_0) > r_1^*$, it follows that $\pi_{\lambda}(p_{\lambda}^{\ast}(m_0), r_1^*) \leq \pi_{\gamma}(p_{\lambda}^{\ast}(m_0), r_1^*)$, and $\pi = \min(\pi_{\lambda}, \pi_{\gamma}) = \pi_{\lambda}$. Hence (13) can be written as: $\pi(p_1, r_1^*) - \pi_{\lambda}(p_{\lambda}^{\ast}(m_0), r_1^*) \geq \pi(p_1, r_1) - \pi(p_{\lambda}^{\ast}(m_0), r_1)$. Combining with equation (12) gives $\pi_{\lambda}(p_{\lambda}^{\ast}(m_0), r_1^*) + \beta J(m_0, p_{\lambda}^{\ast}(m_0)) < \pi(p_1, r_1^*) + \beta J(\min(p_1, m_0), p_1)$. This contradicts the fact that $(m_0, p_{\lambda}^{\ast}(m_0))$ is a steady state of (3). Thus $p_t \geq p_{\lambda}^{\ast}(m_0)$ for all $t$. \[ \square \]

Claim 3. Given $m_0 \in R_2$, the corresponding price trajectory $p_t \geq m_0$ for all $t > 1$.

Proof. We first show that $p_1 = p^*(m_0, p_0) \geq m_0$, hence by induction $p_t^* \geq m_0, \forall t$. Suppose that $p_1 < m_0$; we show that $(m_0, m_0)$ cannot be a steady state of Problem (3), a contradiction. Indeed, $p_1 < m_0$ implies $p_1 \leq \theta m_0 + (1 - \theta) p_0 = r_1$, so $\pi = \min(\pi_{\lambda}, \pi_{\gamma}) = \pi_{\gamma}$. By optimality of $p_1 < m_0$,

$$\pi_{\gamma}(p_1, r_1) - \pi_{\gamma}(m_0, r_1) > \beta(J(m_0, m_0) - J(p_1, p_1)) \geq \beta \Delta J. \quad (14)$$
On the other hand, because $\pi_\gamma(p, r)$ is supermodular, and $r_1 > m_0$, it follows that:

$$\pi_\gamma(p_1, r_1) - \pi_\gamma(m_0, r_1) \leq \pi_\gamma(p_1, m_0) - \pi_\gamma(m_0, m_0).$$

(15)

Combining (14) and (15), we have $\pi_0(m_0) + \beta J(m_0, m_0) < \pi_\gamma(p_1, m_0) + \beta J(p_1, p_1)$. This contradicts $(m_0, m_0)$ being a steady state of (3). We conclude that, if $\underline{m} \leq m_0 \leq \overline{m}$, then $p_t \geq m_0$ for all $t$.

Proof of Lemma 4(ii): From Lemma 3, $(\overline{m}, \overline{m})$ is a steady state of Problem (3). Consider two cases: $p_0 = m_0$ and $p_0 > m_0$.

Case 1: Assume $m_0 = p_0$. Consider the problem: $\tilde{J}(p_{t-1}) = \max_{p_t} \left\{ \pi(p_t, p_{t-1}) + \beta \tilde{J}(p_t) \right\}$. Because the value function in Problem (3) is increasing in its arguments, $J(m_{t-1}, p_{t-1}) \leq \tilde{J}(p_{t-1})$. Equality occurs if $m_{t-1} = p_{t-1}$ for all $t$, i.e. starting at $m_0 = p_0$, the price path is decreasing. The unique steady state of $\tilde{J}$ is $\overline{m}$. Because $\pi(p_t, p_{t-1})$ is supermodular, and starting at $p_0 > \overline{m}$, the optimal price path of $\tilde{J}$ is decreasing to $\overline{m}$. Starting at $p_0 = m_0 > \overline{m}$, the optimal price path of $\tilde{J}$ is feasible for Problem (3) and yields the same value. This is because, at each stage along this path, the minimum price is $m_{t-1} = p_{t-1}$, which implies $\min(p_{t-1}, p_t) = p_t$ and $r_t = p_{t-1}$. Therefore this price path is optimal for Problem (3) and converges to $\overline{m}$. This also implies $m_{t-1} \geq \overline{m}$, as desired.

Case 2: Now assume that $p_0 > m_0$. The following claim proves the desired result.

Claim 4. For $p_0 > m_0 > \overline{m}$, if the optimal price $p_t$ is such that $p_t \leq m_0$, then $p_t \geq \overline{m}$.

Proof. We show that if $p_1$ is such that $p_1 \leq m_0$, then $p_1 > \overline{m}$. Induction further implies $p_t \geq \overline{m}$. Suppose by contradiction that $p_1 = p^*(m_0, p_0) < \overline{m}$, so

$$\pi_\gamma(p_1, r_1) - \pi_\gamma(\overline{m}, r_1) \leq (J(\overline{m}, \overline{m}) - J(p_1, p_1)) \triangleq \beta \Delta J,$$

(16)

where $\pi = \min(\pi_\lambda, \pi_\gamma) = \pi_\gamma$, because $p_1 \leq \theta m_0 + (1 - \theta)p_0 = r_1$. On the other hand,

$$\pi_\gamma(p_1, r_1) - \pi_\gamma(\overline{m}, r_1) \leq \pi_\gamma(p_1, \overline{m}) - \pi_\gamma(\overline{m}, \overline{m}).$$

(17)

by supermodularity of $\pi_\gamma$ and $\theta m_0 + (1 - \theta)p_0 = r_1 > \overline{m} > p_1$. Combining (16) and (17), we have $\pi_0(\overline{m}) + \beta J(\overline{m}, \overline{m}) < \pi_\gamma(p_1, \overline{m}) + \beta J(p_1, p_1)$. This implies that $(\overline{m}, \overline{m})$ cannot be a steady state of Problem (3), a contradiction. We conclude that, if $m_0 > \overline{m}$, then $p_t \geq \overline{m}$.

Proof of Proposition 1 (Step II.2.): It remains to finalize part (c). We already showed, in the proof of Lemma 4(ii), that for $p_0 = m_0 \geq \overline{m}$, prices decrease to the steady state $(\overline{m}, \overline{m})$. Now, we focus on the case $p_0 > m_0$. The next claim ensures that prices eventually fall below $m_0$. 


Claim 5. Starting at \( p_0 > m_0 \geq m \), at some time, \( T \), the optimal price falls below \( m_0 \), i.e. \( p_T \leq m_0 \).

Proof. Suppose by contradiction that the optimal price path is such that \( \{p_t\} > m_0 \). Thus the value function in Problem (3) can be written as: \( J_{m_0}(p_0) = \max_{p_1} \{ \pi(p, r) + \beta J_{m_0}(p_1) \} \). The objective function is supermodular in \((p, r)\) (Lemma 1), so the price path is monotonic, and converges to a steady state. This must be \((m, m)\), which contradicts \( p_t > m_0 \geq m \). We conclude that at some point in time, \( T \), the optimal price is such that \( p_T \leq m_0 \).

Let \( T \) be the first time that the optimal price falls below \( m_0 \). Thus at time \( T \), the value function in Problem (3) can be written as: \( J_{m_0}(p_T) = \max_{p_T \leq m_0} \{ \pi(p, r) + \beta J_{m_0}(p_T) \} \). Claim 4 implies \( p_T > m \). For \( p_0 = m_0 > m \), the price path decreases monotonically to \( m \), and \((m, m)\) is the corresponding steady state. The value function for \( t < T \) is given by the finite horizon model: \( J_{t-1}(p_{t-1}) = \max_{p_t} \{ \pi(p, r) + \beta J_t(p_t) \} \), where \( J_T(p_T) = J_{m_0}(p_T) \). Lemma 1 ensures that the price path is monotone and decreases to \( p_{T-1} \). Note that it cannot be increasing, because then it would have to converge to a steady state above \( m_0 \), contradicting Lemma 3. In summary, starting at \( p_0 > m_0 > m \), the price path decreases until it falls below \( m_0 \), and further down to \( m \).

Proof of Proposition 2: (a) \( \pi(p, r) \) is supermodular in \((p, r)\) by Lemma 1, and \( r = r(\theta) = p + \theta(m - p) \) is decreasing in \( \theta \) for \( m \leq p \). Therefore, \( \pi \) is submodular in \((p, \theta)\) and \( p^*(m, p; \theta) \) is decreasing in \( \theta \). Moreover, because \( \pi \) is increasing in \( r \), and \( r \) is decreasing in \( \theta \), the value function, \( J(m, p; \theta) \), is decreasing in \( \theta \) (Stokey et al. 1989, Thm 4.7). For \( m \geq m, r = p \) because the price path decreases monotonically (Proposition 1). Therefore, \( \pi \), and as a result \( p^*(m, p; \theta) \) and \( J(m, p; \theta) \), are independent of \( \theta \). (b, c) \( p_\lambda^*(m; \theta) \) is decreasing in \( \theta \) and \( \lambda \) because \( p_\lambda^*(m) \geq m \) (for \( m \leq m \)) solves equation (6) the LHS of which is decreasing in \( \theta \), \( \lambda \) and \( p \) (for \( m \leq p \)).