Does Loss Aversion Preclude Price Variation?

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Abstract. In modern retailing, frequent discounts are seemingly at odds with the idea that price variation antagonizes loss-averse consumers and hence diminishes their demand for products and services. We model a monopolist selling a product over time to loss-averse consumers who differ in their sensitivity to gains/losses. Although the market is thus segmented, the firm cannot price-discriminate among consumers based on that sensitivity. We show that charging a long-run constant price may be suboptimal and then derive conditions under which the optimal policy is cyclic (e.g., a periodic markdown policy). These findings establish that loss aversion does not preclude price variation and thereby underscore the importance of incorporating consumer heterogeneity into pricing policies.

Keywords. behavioral pricing, loss aversion, cyclic pricing, markdown management, retailing

1. Introduction

Cyclic pricing, a practice followed in marketing and revenue management, is prevalent in certain industries. Promotional sales account for nearly half of all purchases in US supermarkets (Nielsen, 2009). Retailers often maintain a regular price and offer seasonal sales, holiday (post-Thanksgiving, “Cyber Monday”, etc.) sales, and year-end sales. The year 2016 saw $3.39 billion in Cyber Monday sales in the United States alone (Adobe, 2016), which demonstrates the effectiveness of such pricing policies.

Despite its prevalence and practical success, cyclic pricing seems to be inconsistent with findings in economic psychology (and especially prospect theory) that price variation antagonizes consumers and reduces their demand for products and services. Prospect theory posits that individuals are loss averse and so react more strongly when their payoffs fall short of (than exceed) a reference point. In a pricing context, loss aversion implies that consumers dislike prices that are higher than
expected (perceived surcharges) more than they like prices that are lower than expected (perceived discounts). It follows that consumers are less likely to buy a product or service if its price exceeds expectations (i.e., the reference price). Yet stimulating consumer demand by offering discounts has the negative long-term consequence of eroding the consumer’s reference point, which makes restoring prices to their former levels costly.

We therefore seek to establish whether (or not) loss aversion rules out price variation—and, in particular, cyclic pricing. Evidence is building in the OM literature that loss aversion does not preclude price variation. A demand function subject to strong seasonality effect (Chen et al., 2016), varying consumer shopping schedules (Wang 2016), and demand aggregation (Hu and Nasiry 2017) may lead to price variation even if all consumers are loss averse. We add to this evidence by suggesting an alternative mechanism. We develop a model of a monopolist selling to loss-averse consumers over time. The model has two key ingredients. The first is consumer heterogeneity with respect to perceptions of gain and loss, which we capture by assuming that the market consists of two segments each characterized by different sensitivities to reference effects. This type of heterogeneity has strong support in the empirical and experimental literature, and there have been calls to investigate its normative implications; for a review, see Kopalle et al. (2012) and the references therein. The second is that the firm may use price to shut down the demand from a consumer segment. This is a key role of pricing in managing demand especially when the firm cannot price discriminate between the segments.

Considerable evidence supports consumers’ heterogeneity in gain/loss perceptions. Bell and Lattin (2000) use scanner panel data to measure loss aversion in nondurable grocery products and find that using a single-segment model tends to significantly overestimate loss aversion. Moon et al. (2006) estimate the reference effect in a consumer product category and find that the market consists of three segments: a segment without gain/loss sensitivity, a segment with a memory-based (i.e., internal) reference point, and one with a stimulus-based (i.e., external) reference point. They show that price sensitivities are significantly different across the segments. Dayaratna and Kannan (2012) provide an empirical estimation for their memory-based reference price reformulation using a three-segment latent class model across which gain/loss sensitivities differ significantly. We further refer to Neumann and Böckenholt (2014) for a recent comprehensive review.

In our model, each segment consists of a continuum of consumers with uniformly distributed valuations (i.e., willingness to pay or WTP). This implies that the ‘base demand’ of each segment, that is, the demand in absence of behavioral considerations, is linear. We allow one segment (which we call “segment 1”) to have (stochastically) higher WTP. We show that if consumers in the segment with lower WTP (i.e., “segment 2”) exhibit a strong reaction to perceived gains, then the firm’s optimal pricing policy is a cyclic one. A common form of cyclic pricing is markdown pricing, whereby—once in each cycle—the firm targets segment-2 consumers by offering a discount. If the discount is set optimally, then its stimulation effect on segment-2 demand dominates its dilution.
effect on the profit margin of products sold to segment-1 consumers. The cycle length and price range of the optimal cyclic policy are volatile and non-monotone in the model’s behavioral parameters and especially the strength of gain/loss perception. Finally, we show that ignoring consumer heterogeneity in gain/loss perception leads to constant pricing, which significantly underperforms the optimal pricing policy.

We use an exponentially weighted average of past prices as the reference point formation mechanism in our base model. This mechanism is recursive: the new reference point is a function of the previous reference point and the previous piece of information. Recursivity makes a dynamic model amenable to analytical analysis and hence exponential smoothing is widely applied in the dynamic pricing and revenue management literature (see e.g., Popescu and Wu 2007). However, from an experiment, Baucells et al. (2011) observe that reference prices are not recursive. Our key results do not depend on the particular assumption of exponential smoothing or the recursivity of the reference point. Our results generalize, under mild conditions on the reference point formation mechanism, to contexts in which consumers may remember only a few price points or follow a peak-end rule.

Consumers in this model are loss averse, and our assumptions ensure that the overall demand (of each consumer segment) is also more sensitive to losses than to gains. In that sense, both the research question and our approach to studying it differ from the literature. A few papers have established the optimality of cyclic (or “hi-lo”) pricing under reference effects (Greenleaf 1995, Kopalle et al. 1996, Popescu and Wu 2007, Hu et al. 2016, Chen et al. 2016, Wang 2016). However, these papers (except for Chen et al. 2016 and Wang 2016) assume that at least some consumers are gain seeking—in other words, they are actually more sensitive to gains (when the firm charges a price below the reference price) than to losses (when the firm charges a price higher than the reference price). In this case, the firm applies hi-lo pricing as follows: after increasing the reference point by charging a series of high prices, it then exploits that higher reference point by offering a discount that significantly boosts the demand from gain-seeking consumers.

Chen et al. (2016) show numerically that a strong seasonality effect on the demand function of loss-averse consumers causes a cyclic pricing pattern. Wang (2016) studies the dynamic pricing problem of a monopolist in a market of consumers who have different shopping schedules. In his model, consumers’ reference points are based only on the prices they encounter when visiting the store; therefore, consumers with different shopping schedules have different reference points. Wang proves that the firm’s optimal policy for loss-neutral consumers is cyclic, and he demonstrates numerically that the same insight holds also for loss-averse consumers. Our paper complements his work by accounting for consumer heterogeneity when pricing for reference-dependent consumers. However, the results reported here contradict his claim that, “if customers have homogenous arrival times, then even if they have different demand functions and memory factors, the optimal pricing policy must be an asymptotically constant one” (Wang 2016, p. 291). We confirm the optimality
of cyclic pricing, in the absence of heterogeneous arrival times, by incorporating the heterogeneity in consumers’ sensitivity to reference effects. We discuss Wang (2016) more thoroughly in Section 2.

Besbes and Lobel (2015) study the dynamic pricing problem of a monopolist in a market of strategic consumers who are heterogeneous with respect to both willingness to pay and willingness to wait. These authors assume the monopolist commits to a price sequence and then show that the optimal price policy is cyclic with length no more than twice the maximum of the customer population’s willingness to wait.

Hu and Nasiry (2017) argue that behavioral biases, such as loss aversion, are individual-level phenomena and so need not be “inherited” by such aggregate variables as market demand. That is, it may be that aggregate demand is more sensitive to gains than to losses even though all consumers in the market are loss averse. These authors suggest that, instead of adding a gain/loss component to a conventional demand function, it would be preferable to derive individual demand subject to loss aversion and then aggregate it while accounting for the consumer population’s valuation heterogeneity. Hu and Nasiry show that gain/loss preferences are inherited by the aggregate demand function if and only if consumer valuations are distributed uniformly. The consumers in our model are loss averse, and consumer valuations in each of the two segments are uniformly distributed. This implies that the overall demand in each market segment is more sensitive to losses than to gains. Yet the two segments do still have different sensitivities, so if the firm could price-discriminate then the price path would converge to a steady state (Popescu and Wu 2007; Hu and Nasiry 2017). This type of price discrimination is not plausible in the retail context, where each product carries only one price at any given time. We demonstrate that, when the firm cannot discriminate, its optimal pricing policy may not admit a steady state and could, in particular, be cyclic.

Periodic markdowns as a pricing and demand management method have been extensively studied in the operations management and economics literature. Özer and Zheng (2015) show that a firm may prefer markdown pricing to an “everyday low price” strategy when consumers are prone to regret and probability weighting. Yin et al. (2009), Mersereau and Zhang (2012), Ovchinnikov and Milner (2012), and Cachon and Feldman (2015), among others, study the optimality of markdown pricing in the presence of strategic consumers. Our own focus is not on markdown policies per se, and neither do we incorporate strategic consumer behavior. Instead, we endeavor to prove that loss aversion—even at the aggregate demand level—does not rule out price variation if we account for the heterogeneity in consumer perceptions of gains and losses.

From a managerial perspective, our work suggests that the effect of consumer loss aversion on pricing policies is mediated by consumer heterogeneity in the strengths of gain/loss perceptions. In particular, attempts to fit a single gain/loss parameter to market demand data will result in adopting suboptimal pricing policies (e.g., uniform instead of cyclic policies) and profit loss.
2. Model

In this section we introduce the general setup of our pricing model. A monopolist sells a product to a market of loss-averse consumers over an infinite horizon. We assume that the market consists of two segments \( i \in \{1, 2\} \). There is a continuum of \( a_i \) consumers in segment \( i \) whose valuations are distributed uniformly in the interval \([0, a_i/b_i]\).\(^1\) The parameter \( a_i/b_i \) is the maximum WTP in segment \( i \). The firm cannot price-discriminate between the two segments, so the consumers in each segment encounter the same price at time \( t \). In deciding whether to buy at that time, consumers compare \( p_t \) (the price at time \( t \)) with a reference point \( r_t \). A price higher (resp. lower) than the reference point is perceived as a surcharge (resp. discount).

We assume that the reference price is an exponentially weighted average of past prices:

Assumption 1. \( r_t = \theta r_{t-1} + (1 - \theta)p_{t-1} \) for some \( \theta \in (0, 1) \).

The parameter \( \theta \) is a memory parameter that captures how well consumers recall past prices. Memory-based reference prices are popular in both empirical and modeling research in the marketing and revenue management literature (see e.g. Greenleaf, 1995; Kopalle et al., 2012; Popescu and Wu, 2007; for a review, see Mazumdar et al., 2005). In Section 3 we will show that our key results generalize to a broader set of reference point formation mechanisms of which exponential smoothing is a special case.

Segment \( i \)'s demand function at time \( t \) is

\[
D_i(p_t, r_t) = (a_i - b_i p_t + \gamma_i (r_t - p_t)^+ - \lambda_i (p_t - r_t)^+) + ,
\]

where \( \gamma_i \) and \( \lambda_i \) are the demand function’s sensitivity to discounts and surcharges, respectively.

We choose uniform consumer valuation distributions for two reasons. First, this assumption implies that the ‘base’ demand function of segment \( i \), \( D_i(p_t) = a_i - b_i p_t \) is linear (see e.g. Phillips, 2005, Chapter 3). Linear demand functions are commonly applied and perform sufficiently well in modelling demand in dynamic pricing settings (Besbes and Zeevi, 2015).

Second, because consumer valuations in segment \( i \) are distributed uniformly, their biases toward gains and losses are inherited by the segment’s aggregate demand (Hu and Nasiry, 2017). In other words: if all individual consumers in segment \( i \) are loss averse or loss neutral then the overall demand of segment \( i \) is, respectively, more sensitive to losses (\( \lambda_i > \gamma_i \)) or equally sensitive to losses and gains (\( \lambda_i = \gamma_i \)). This property is important because our goal is to investigate whether loss aversion at the aggregate demand level precludes price variation. If loss aversion is present at the individual level but absent at the aggregate level, which Hu and Nasiry show is possible, then the optimal price policy may not admit a steady state (Hu and Nasiry, 2017).

If there is only one segment in the market then, in the long run, the firm’s optimal policy

\(^1\)It may be that \( a_1 = a_2 \) and \( b_1 = b_2 \).
converges to a steady state. The reason is that the overall demand function is more sensitive to losses
than to gains (Fibich et al., 2003; Popescu and Wu, 2007; Hu and Nasiry, 2017). However, we show
that this generalization no longer holds when there are two market segments with heterogeneous
sensitivities to losses and gains. We shall therefore assume without loss of generality that \( a_2/b_2 \leq
a_1/b_1 \).\(^2\) Obviously, at prices above \( a_1/b_1 \) the demand from both segments is zero unless customers
perceive substantial gains. Because that is unlikely to happen,\(^3\) we assume that \( p_t \) (and hence \( r_t \))
is no larger than \( a_1/b_1 \). This assumption however allows the firm to shut down the demand from
segment 2 by charging prices above the “choke price” of segment 2, i.e., \( p_t \geq a_2/b_2 \). Thus our
setup differs from that of Wang (2016), who assumes the firm can charge only those prices at which
the demand from every segment is positive. That assumption rules out a key function of prices in
managing demand, and it is the main reason why cyclic pricing cannot be optimal in his setup when
arrival times are homogeneous. Popescu and Wu (2007) make the same assumption and prove that,
in the long run, it is optimal for the firm to charge a uniform price if there are multiple market
segments with loss-averse consumers. Cyclic pricing will emerge in their model of heterogeneous
consumers if the monopolist can price out one or more segments.

In this paper, we mainly focus on the case in which \( \gamma_1 = \lambda_1 = 0 \) and \( \gamma_2 \leq \lambda_2 \). That is, demand
from segment 1 is not subject to reference effects whereas demand from segment 2 is either more
sensitive to losses or equally sensitive to losses and gains. Assumption \([2]\) facilitates the analytics
in our paper. In Section 4.3 our results generalize to a setting where both segments are subject to
reference effects. In summary, we make the following assumption in this section.

**Assumption 2.** We have \( a_2/b_2 \leq a_1/b_1, \gamma_1 = \lambda_1 = 0, \gamma = \lambda \leq \lambda_2 \), and \( p_t, r_t \in [0, a_1/b_1] \).

Moon et al. (2006) provide empirical support for this assumption. They estimate consumers’
responses to price changes using a dataset on the toilet tissue purchase records. They find that some
consumers are not prone to reference effects (i.e., \( \lambda_1 = \gamma_1 = 0 \)), while others are loss averse (i.e.,
\( \gamma_2 < \lambda_2 \)). Moreover, loss-averse consumers constitute a larger proportion of the market \( (a_2 > a_1) \)
and are more price sensitive \( (b_2 > b_1) \).

By Assumption \([2]\) the demand of segment 1 is \( D_1(p_t) = a_1 - b_1p_t \) and that of segment 2 is
\( D_2(p_t, r_t) = (a_2 - b_2p_t + \gamma(r_t - p_t)^+ - \lambda(r_t - r_t)^+) \). Let \( \Pi_1(p_t) = p_tD_1(p_t) \) and \( \Pi_2(p_t, r_t) = p_tD_2(p_t, r_t) \) denote the single-period revenue functions corresponding to segments 1 and 2, respectively. The aggregate demand from both segments is \( D(p_t, r_t) \triangleq D_1(p_t) + D_2(p_t, r_t) \), and the corresponding single-period revenue is \( \Pi(p_t, r_t) \triangleq p_tD(p_t, r_t) \).

\(^2\) Mathematically, the valuation distribution of segment 1 stochastically dominates that of segment 2.

\(^3\) If the reference point is arbitrarily large, then the demand from either segment may be positive for prices
above \( a_1/b_1 \). Yet such large reference price values can occur only if past prices are consistently above \( a_1/b_1 \)—a pricing
policy that would be impractical for the firm. Even if the firm charges such prices, they are unlikely to be ‘assimilated’
into the consumers’ reference point; see Narasimhan et al. (2005) p. 365-366.
2.1. Myopic Pricing Policy

We first study the firm’s myopic pricing policy, under which the firm focuses on maximizing current-period revenue and ignores this strategy’s possible consequences on the firm’s long-term revenues. That is, given the reference price \( r \) in the current period, the firm uses the price \( p_m(r) = \text{argmax}_p \{ \Pi(p, r) \} \). As we shall see, the myopic pricing policy has an analytical solution and thus provides useful insights into the benefit of cyclic pricing.

To give an intuition for why myopic firms may prefer cyclic pricing, consider a periodic markdown policy: in a pricing cycle of \( n \) periods, the firm targets only segment 1 for \( n - 1 \) periods and then offers a discount to “skim” segment 2 once in a cycle. More precisely, from period 1 to \( n - 1 \), the firm charges a regular price \( \frac{a_1}{2b_1} \), which is maximizing the revenue \( \Pi_1(p) \) from segment 1 and ignores segment 2. In period \( n \), the firm charges a price (which is characterized in Proposition 6 in Appendix A.1) that is less than \( \frac{a_1}{2b_1} \) to appeal to segment 2 customers and meanwhile optimize its revenue from both segments \( \Pi_1(p) + \Pi_2(p, r_n) \). Then from period \( n + 1 \) to \( 2n - 1 \), the same regular price is used and so on. Seasonal sales are an example of periodic markdown policies commonly employed in practice.

To understand why periodic markdown can be optimal for a myopic firm, we track how the reference price of segment 2 customers changes in a pricing cycle. In period 1, the reference price is relatively low. To make a profit from segment 2, the firm has to set a low price and forgo a substantial amount of revenue from segment 1. Therefore, the optimal myopic pricing policy is to set \( p_m(r_1) = \text{argmax}_p \Pi_1(p) = \frac{a_1}{2b_1} \) that prices out segment 2. Since this price is higher than the reference price \( r_1 \), by Assumption 1 the reference price in the next period \( r_2 \) increases slightly due to customers’ memory in period 1. However, \( r_2 \) is still not large enough and the firm chooses to price out segment 2 in period 2 as well. The same happens until period \( n \), in which customers in segment 2 have a sufficiently large reference price \( r_n \) based on their memory from periods 1 to \( n - 1 \). It is now optimal for the firm to set a low price so that segment 2 customers perceive gains and the boosted demand from segment 2 outweighs the lost revenue from segment 1 as the price deviates from \( \frac{a_1}{2b_1} \). In period \( n + 1 \), however, the reference price drops due to the discount in period \( n \), and the pricing cycle starts over again. Proposition 6 in Appendix A.1 provides the condition under which a cycle-\( n \) periodic markdown is optimal for a myopic firm.

Having explained the intuition, we study the analytical solution for the myopic pricing policy, i.e., maximizing the revenue function \( \Pi(p, r) \) for given \( r \). In general, \( \Pi(p, r) \) is piecewise quadratic, non-smooth, and non-concave, making the optimization quite challenging. It has one of the follow-
Segment 2 is priced out
\[ \Pi(p, r) = \Pi_1(p) = p(a_1 - b_1p) \]
Segment 2 perceives gains
\[ \Pi(p, r) = p((a_1 + a_2) - (b_1 + b_2)p + \gamma(r - p)) \]
Segment 2 perceives losses
\[ \Pi(p, r) = p((a_1 + a_2) - (b_1 + b_2)p + \lambda(r - p)) \]

Therefore, we have to identify \(p_m(r)\) from potential local maxima \(\{\frac{a_1}{2b_1}, p^\gamma(r), p^\lambda(r), r\}\), where we define \(p^\gamma(r) \triangleq \frac{a_1 + a_2 + \gamma r}{2(b_1 + b_2 + \gamma)}\), \(p^\lambda(r) \triangleq \frac{a_1 + a_2 + \lambda r}{2(b_1 + b_2 + \lambda)}\). While the first three local maxima in the set correspond to the three quadratic functions above, the last maximum is produced by the non-smoothness at \(p = r\). Next we compare the local maxima and derive conditions under which one of them becomes the global maximum. This results in Proposition I.

Proposition 1. One and only one of the following cases emerges:

1. If \(\bar{r}^0 \leq r^\lambda\), then (i) \(p_m(r) = a_1/2b_1\) for \(r \in [0, \bar{r}^\lambda]\), (ii) \(p_m(r) = p^\lambda(r)\) for \(r \in (\bar{r}^\lambda, r^\gamma]\), (iii) \(p_m(r) = r\) for \(r \in (r^\lambda, r^\gamma]\), and (iv) \(p_m(r) = p^\gamma(r)\) for \(r \in (r^\gamma, a_1/b_1]\).

2. If \(\bar{r}^0 \in (r^\lambda, r^\gamma]\), then (i) \(p_m(r) = a_1/2b_1\) for \(r \in [0, \bar{r}^0]\), (ii) \(p_m(r) = r\) for \(r \in (\bar{r}^0, r^\gamma]\), and (iii) \(p_m(r) = p^\gamma(r)\) for \(r \in (r^\gamma, a_1/b_1]\).

3. If \(\bar{r}^0 > r^\gamma\), then (i) \(p_m(r) = a_1/2b_1\) for \(r \in [0, \bar{r}^\gamma]\) and (ii) \(p_m(r) = p^\gamma(r)\) for \(r \in (\bar{r}^\gamma, a_1/b_1]\).

In each case, the myopic policy \(p_m(r)\) is piecewise linear and discontinuous. Take case 1 for an example. When the reference price is low, the firm simply maximizes the revenue from segment 1 and prices out segment 2 (case (i)). As the reference price increases, segment 2 becomes more profitable and the firm targets both segments. It may cause segment 2 to perceive losses (case (ii)), or gains (case (iv)), or neither (case (iii)) under the myopic policy. Case 2 and 3 are similar to case 1, while one or more sub-cases become non-existent. Proposition I is illustrated in Figure I.

2.1.1. Steady States and Cyclic Pricing

This section focuses on the behavior of the myopic pricing policy and investigates the conditions under which it is cyclic—in other words, when it does not admit a steady state. Given a reference price \(r_t\) at time \(t\), the myopic firm charges \(p_m(r_t)\) (as specified in Proposition I) and so the reference price at time \(t+1\) becomes \(r_{t+1} = \theta r_t + (1 - \theta)p_m(r_t)\). A myopic steady-state price is a price from which the myopic firm has no incentive to deviate; it is a price \(\hat{r}\) that solves \(p_m(\hat{r}) = \hat{r}\). If \(r_t = \hat{r}\) at time \(t\), then \(p_m(r_t) = r_t\) and \(r_{t+1} = \theta r_t + (1 - \theta)p_m(r_t) = r_t\), which implies that \(r_t\) and the optimal myopic price \(p_m(r_t)\) remains \(\hat{r}\) over time.
We can use Figure 1 to illustrate the existence of steady states. The steady state corresponds to the intersection of an identity map (dotted line in the figure) and the function $p_m(\cdot)$. Therefore, case 1 and case 2 in Proposition 1 (which correspond to panels (a) and (b) of Figure 1) admit a range of steady states.

In case 3 of the proposition there are two possibilities. If $a_1/2b_1 > \bar{r}^\gamma$, then there is no steady state (panel (c) in the figure). But if $a_1/2b_1 \leq \bar{r}^\gamma$ then there is a single steady state, $a_1/2b_1$ (see panel (d)). The following proposition formally characterizes the myopic steady states.

**Proposition 2.**

(i) If $\bar{r}^0 \leq r^\lambda$, then any $r \in [r^\lambda, r^\gamma]$ is a steady state.

(ii) If $\bar{r}^0 \in (r^\lambda, r^\gamma]$, then any $r \in [\bar{r}^0, r^\gamma]$ is a steady state.

(iii) If $\bar{r}^0 > r^\gamma$ and $\bar{r}^\gamma \geq a_1/2b_1$, then there is one steady state: $a_1/2b_1$.

(iv) If $\bar{r}^0 > r^\gamma$ and $\bar{r}^\gamma \in (r^\gamma, a_1/2b_1)$, then there is no steady state.
Two observations follow from this proposition. First, the steady states from cases (i) and (ii) are essentially different from the steady state in case (iii). In (i) and (ii), the firm sells to both segments in the steady state whereas segment 2 is priced out in case (iii). Second, the emergence of a cyclic pricing policy (case (iv)) depends on $\gamma$ but not on $\lambda$, because $\bar{r}_0$, $r^{*}$ and $\bar{r}^{\gamma}$ are not functions of $\lambda$. As $\gamma \to \infty$, we have $r^{\gamma} = O(\gamma^{-1})$ and $\bar{r}^{\gamma} = O(\gamma^{-1/2})$. These imply that, when the sensitivity to perceived gains increases, the condition in case (iv) is more likely to be met; in that event, hi-lo pricing becomes preferable (from the firm’s perspective) to uniform pricing. Our next result shows that if a steady state exists then, in the long run, the paths of the optimal myopic price charged by the firm and the reference price of customers converge to that steady state—although the convergence is not monotonic.

**Proposition 3.** (i) For all $r_0 \in [0, a_1/b_1]$, if the myopic pricing policy admits steady states (cases (i), (ii), (iii) in Proposition 2), then $r_t$ and $p_m(r_t)$ converge to the steady state. (ii) If there is no steady state (case (iv) in Proposition 2), then $r_t$ and $p_m(r_t)$ are cyclic in the long run.

Figure 1 plots the reference price path. If the reference price at time $t$ is $r_t$ then, at time $t + 1$, the reference price is $r_{t+1} = \theta r_t + (1 - \theta)p_m(r_t)$. In all panels of Figure 1 the horizontal line from $(r_t, \theta r_t + (1 - \theta)p_m(r_t))$ intersects the identity map (the 45° line) at $(r_{t+1}, r_{t+1})$. The bold dots in Figure 1 indicate how $r_t$ evolves over time. The reference price clearly converges to a steady state in panels (a), (b), and (d) of the figure whereas, in panel (c), the path cycles.

However, the cycle length is not monotone in any of the parameters. Instead it displays the so-called bifurcation phenomenon of dynamical systems theory (see Rajpathak et al., 2012 and the references therein): between two regions of the parameter space corresponding to cycle lengths $n_1$ and $n_2$, there exists a region with cycle length $n_1 + n_2$. This behavior implies that the cycle length is extremely discontinuous in the parameters; the phenomenon is consistent with the fact that widely different promotion patterns are observed for products that are but slightly differentiated.

It is worth mentioning that, unlike in Popescu and Wu (2007) and Nasiry and Popescu (2011), convergence to the steady state is not order preserving; that is, a larger $r_0$ does not guarantee a larger steady state. For example, in Figure 1(a), if $r_0 = r^\lambda$, then $r_0$ is already a steady state; if $r_0$ is set as illustrated by the dots in that panel and thus less than $r^\lambda$, then $r_0$ reaches a steady state that is greater than $r^\lambda$. This is because the evolution of the reference price is discontinuous and non-monotone in $r$.

### 2.2. Optimal Dynamic Pricing

In this section, we study the firm’s dynamic pricing problem. This problem differs from the myopic pricing problem, in which the firm considers only current-period profits. Given the initial reference price $r_0$, the firm maximizes its infinite-horizon problem as $V(r_0) = \max_{p_t} \sum_{t=0}^{\infty} \beta^t \Pi(p_t, r_t)$, where $\beta < 1$ is the discount factor. Since $0 \leq p_t \leq a_1/b_1$, it follows that per-period revenue is bounded;
hence the value function is the unique solution to the Bellman equation

\[ V(r) = \max_{p \in [0,a_1/b_1]} \{ \Pi(p, r) + \beta V(\theta r + (1 - \theta)p) \}. \]  

(2)

Let \( p^*(r) \) denote the optimal pricing policy. In general, the Bellman equation (2) does not admit a closed-form solution. To obtain structural results, we first derive a performance bound—that is, an upper bound for the revenue gap between the myopic and the optimal pricing policies; then we can show that the optimal dynamic policy does not admit steady states when the gains-related reference effect is strong.

2.2.1. A Performance Bound

We use \( V_m(r) \) to denote the discounted revenues corresponding to the myopic pricing policy when the current reference price is \( r \). By definition, \( V_m(r) = \Pi(p_m(r), r) + \beta V_m(\theta r + (1 - \theta)p_m(r)) \). We can obtain the following result.

**Proposition 4.**

\[ V(r) - V_m(r) \leq \frac{\beta(1 - \theta)\lambda a_1^2}{(1 - \beta)(1 - \beta \theta)b_1^2}. \]

Two key observations can be made with regard to this theorem. First, the bound approaches zero when \( \beta \to 0 \) or \( \theta \to 1 \). That is to say: the less weight is given to future revenues (i.e., the more myopic is the firm), the tighter is the bound. The myopic case is equivalent to \( \beta = 0 \) in (2). When \( \theta = 1 \), consumers never update their reference prices. Thus the dynamic problem is equivalent to a sequence of identical single-period pricing problems and so the myopic pricing policy is optimal.

Second, the performance bound increases with the loss-aversion coefficient \( \lambda \). This outcome follows because a higher \( \lambda \) increases the weight of future revenues in the trade-off between them and current revenues. The myopic pricing policy completely ignores the effect of current pricing on future revenues; as a consequence, it is significantly outperformed by optimal dynamic pricing when \( \gamma \) or \( \lambda \) is large.

2.2.2. Cyclic Pricing and Reference Effects

We now establish that, if the reference effect for perceived gains \( \gamma \) is sufficiently large, then a long-run constant price (steady state) is never optimal in the dynamic case. This result is at odds with Theorems 2 and 4 in [Popescu and Wu (2007)](Popescu and Wu (2007)), which state that there always exists a steady state for loss-neutral or loss-averse customers. That is, no matter how strong reference effects may be, the firm should avoid hi-lo pricing and instead charge a constant long-run price if consumers are loss-neutral or loss-averse. Their results differ because their model does not allow a market segment to be priced out. So even though the perceived gain may be substantial when using hi-lo
pricing, the perceived loss is also large. In the Popescu and Wu model, then, the negative effects of these losses dominate the positive effects of gains and hence adopting a constant price is always optimal in the long run.

In contrast, our model considers the more realistic case in which gain/loss preferences are heterogeneous. The loss effect is capped because the worst-case scenario is to have zero demand from segment 2. The firm can use a regular high price that shuts down the demand from segment 2, and focus only on segment 1, yet occasionally reap some segment-2 revenue when the reference price of those consumers is sufficiently high. Such a strategy is practical and frequently observed when the market is segmented but the firm cannot price-discriminate. This insight is formalized in our next result.

**Proposition 5.** When $\gamma$ is sufficiently large, the optimal dynamic pricing policy that solves (2) does not have any steady states.

By Assumption 2, we must have $\lambda \geq \gamma$ so $\lambda$ is also large for large $\gamma$. Nevertheless, it is the magnitude of $\gamma$ that leads to cyclic pricing regardless of $\lambda$. This may seem counterintuitive given that, when the firm charges a high price, loss aversion ($\lambda > \gamma$) restricts the revenue earned from segment 2; hence a sufficiently large $\lambda$ should discourage the firm from hi-lo pricing and lead to uniform pricing. However, this is not the case. There is indeed very little demand from segment 2 when the firm employs hi-lo pricing and charges a high price. The revenue in this phase is mainly generated from segment 1, whose demand is not subject to reference effects. In other words: no matter how large $\lambda$ is, its negative effect on demand in the high-price phase is limited because the demand from segment 2 cannot be less than zero. On the other hand, the profitability of hi-lo pricing (as compared with uniform pricing) is reflected in the price-cut phase—that is, when the price changes from “hi” to “lo”. The more sensitive the demand of segment 2 to perceived gains, the more profitable hi-lo pricing will be. It follows that the emergence of a steady state depends mainly on $\gamma$. This is consistent with the discussion following Proposition 2.

### 3. A General Reference Point Formation Mechanism

So far we have assumed that consumers’ price expectation in each time period is an exponentially weighted average of past prices. This framework lends itself to analytical analysis and is widely applied in pricing literature. Under this assumption, the reference point formation mechanism is recursive, that is, the new reference point is a function of the previous reference point and the previous piece of information [Baucells et al. (2011)]: $r_t = f(r_{t-1}, p_{t-1})$. Exponential smoothing also implies implicitly that consumers recall all past prices paid for a product.

In this section, we introduce a general reference price formation mechanism of which exponential smoothing is a special case. Our goal is to show that under mild conditions on this general
mechanism, our key insights on a monopolist’s pricing policy (i.e., Proposition 5) continue to hold. That is, cyclic pricing is optimal when the heterogeneity in the reference effects of the two segments is sufficiently large.

Given a sequence of prices, \( p_t \) and the initial reference price \( r_0 \), consider an array of weights \( w_{t,i} \in [0,1] \), where \( t = 0,1, \ldots \) and \( i = -1,0, \ldots, t-1 \), and \( \sum_{i=-1}^{t-1} w_{t,i} = 1 \). The reference price in period \( t \) then is:

\[
r_t = w_{t,-1} r_0 + \sum_{i=0}^{t-1} w_{t,i} p_i.
\]  

In other words, the reference price in period \( t \) is a weighted average of all past prices and the initial reference price. Since we do not specify \( w_{t,i} \), it is more general than most reference formation models in the literature.

As we shall see, the reference point in (3) encompasses exponential weighting, the model proposed in [Baucells et al. 2011] (see their Equation (7)), and extrapolative expectations where the reference price depends on most recent (e.g., the last two) prices; see e.g., [Jacobson and Obermiller 1990]. Moreover, Equation (3) is not necessarily recursive.

To proceed, we define two properties for a reference point formation mechanism: retentiveness and asymptotic obliviousness.

**Definition 1.** The reference point in (3) is \((K, \delta)\)-retentive if there exist \( K \in \mathbb{Z}_+ \) and \( \delta > 0 \) such that for all \( t \geq 1 \), we have \( \sum_{i=(t-K)}^{t-1} w_{t,i} \geq \delta \).

If a reference formation process is \((K, \delta)\)-retentive, then the customer always puts a positive weight on the most recent \( K \) prices, regardless of \( t \). Retentiveness is not restrictive, as the choice of \( K \) and \( \delta \) is free. Customers with a strong memory may have a large \( K \) and a small \( \delta \). For example, a customer may have a 1/3 weight on the prices in the last 10 periods and the remaining 2/3 on the historical prices more than 10 periods ago. For forgetful customers, we may have a small \( K \). For example, a customer forming her reference price based on the average of the prices in the last three periods is \((3, 1)\)-retentive.

**Definition 2.** The reference point in (3) is asymptotically oblivious, if for all \( k \geq 1 \), we have \( \lim_{t \to \infty} \sum_{i=-1}^{k \wedge (t-1)} w_{t,i} = 0 \).

Asymptotic obliviousness implies that customers’ memory of the prices in the first few periods fades away in the long run. If a reference formation process is not asymptotically oblivious, then the customer always assigns some weight to the prices in the first few periods. As a result, even if the firm charges a constant price \( p \) in the long run, the reference price may not converge to \( p \) as \( t \to \infty \). This is unlikely to happen in practice.

**Example 1** (Exponentially-smoothed adaptive expectations process). The reference price model we use in Section 2 has a recursive form \( r_t = \theta r_{t-1} + (1-\theta)p_{t-1} \). This is a special case of the formation
mechanism in (3) as we can write \( r_t = (1 - \theta) \sum_{i=0}^{t-1} \theta^{t-1-i} p_i + \theta^t r_0 \). It is straightforward to observe that this reference point formation mechanism is \((1, 1 - \theta)\)-retentive and asymptotically oblivious.

**Example 2 (Extrapolative expectations).** In this framework, customers’ reference point is a weighted average of \( k \) most recent prices: \( r_t = \sum_{i=1}^{k} a_i p_{t-i} \) and \( \sum_{i=1}^{k} a_i = 1 \). For instance, the model in Jacobson and Obermiller (1990) is a special case with \( k = 2 \). Clearly, this reference price formation mechanism is \((k, 1)\)-retentive and asymptotically oblivious. However, it is not recursive.

Next we analyze the optimal dynamic policy for the firm given the reference point formation in (3). The firm’s optimal pricing policy \( \{p_t\}_{t=0}^{+\infty} \) solves the following optimization problem:

\[
\max_{\{p_t\}_{t=0}^{+\infty}} \sum_{t=0}^{+\infty} \beta^t \Pi(p_t, r_t)
\]

subject to

\[
\begin{align*}
\ r_t &= w_{t-1} r_0 + \sum_{i=0}^{t-1} w_{t,i} p_i, \\
\ p_t &\in [0, a_1/b_1], \ t = 0, 1, \ldots
\end{align*}
\]

Because the reference formation mechanism in (3) does not necessarily have a recursive form, the optimization problem cannot be transformed into a dynamic program and is thus analytically intractable. Nevertheless, we are able to characterize its cyclic behavior in Theorem 1 using a perturbation approach. The result applies to Examples 1 and 2.

**Theorem 1.** If the reference formation mechanism in (3) is \((K, \delta)\)-retentive and asymptotically oblivious, then for a sufficiently large \( \gamma \), the optimal pricing policy \( p^*_t \) for \( t = 1, 2, \ldots \) does not admit a steady state. That is, there exists \( \epsilon > 0 \) such that for all \( T \geq 1 \), \( \max_{t \geq T} p^*_t - \min_{t \geq T} p^*_t > \epsilon \).

Retentiveness and asymptotic obliviousness are not restrictive conditions and the mechanism in (3) encompasses widely-applied reference points. Thus, Theorem 1 establishes the general validity of cyclic pricing when the low-valuation segment has sufficiently strong reference effects.

We finally remark that some reference point structures are not covered by (3). Nonetheless, our key insights may generalize to such structures as well. For example, the ‘peak-end rule’ is a recursive reference point formation mechanism which assumes consumers anchor on the most recent and the minimum prices paid for a product (Nasiry and Popescu 2011), or \( r_t = \theta m_{t-1} + (1 - \theta) p_{t-1} \) where \( m_{t-1} = \min(m_{t-2}, p_{t-1}) \). We show in Appendix A.2 that with the peak-end rule, and for sufficiently large \( \gamma \), the firm’s optimal policy does not admit a steady state.

### 4. Numerical Studies

In this section, we conduct several numerical studies. We first compare the optimal dynamic pricing policy (which may be cyclic) to a uniform price policy in which the firm charges an optimally-set
constant price over time. A uniform price policy could arise if the firm ignores heterogeneity and decides to charge a constant price to loss-averse consumers. We show that this policy can lead to a substantial revenue loss (as high as 46%) compared to cyclic pricing.

In the second study, we use parameters estimated by Moon et al. (2006) and Dahana and Terui (2006) to compute the optimal pricing policy. Our goal in this study is to show that cyclic pricing is optimal for empirical estimates of the key parameters in our model. In the third study, we relax Assumption 2 so that both segments are loss-averse. This example further extends the insight derived in Theorem 1: cyclic pricing is optimal if the loss-effect in the high-valuation segment (segment 1) is much lower than the gain-effect in the low-valuation segment (segment 2), i.e., \( \gamma_2 \gg \lambda_1 \), while \( \lambda_2 \) and \( \gamma_1 \) are not critical for the emergence of cyclic pricing.

4.1. **What Is the Revenue Loss If the Firm Ignores Heterogeneity?**

We built the numerical example under Assumptions 1 and 2. In particular, we set \( a_1 = 1, b_1 = 0.3, a_2 = 4, b_2 = 2, \beta = 0.99 \) and \( r_0 = 0 \). We let \( \lambda = \gamma \) (i.e., consumers in segment two are gain/loss neutral) and vary \( \gamma \). This is for simplicity since by Proposition 2 and the discussion following Theorem 5, \( \lambda \) does not significantly affect cyclic pricing behavior. To solve the Bellman equation numerically, we discretize the state \( r \) and use value iteration and linear interpolation to compute the optimal value function \( V(r) \) and the optimal pricing policy \( p^*(r) \). Table 1 demonstrates the structure of the optimal pricing policy for various combinations of \( \gamma \) and \( \theta \). Table 2 compares the optimal pricing policy with the optimal uniform pricing policy. A cycle length of 1 implies that \( p^*(r_t) \) and \( r_t \) converge to a steady state.

When combined with the previous analysis, the numerical results allow us to make three conclusions. First, cyclic behavior tends to emerge for sufficiently large \( \gamma \)—a dynamic that is consistent with Proposition 5 and Theorem 1. In the numerical example, a long-run uniform pricing policy is optimal only for \( \gamma \leq 0.4 \) (with one exception \( \gamma = 0.6 \) and \( \theta = 0.9 \)), which is roughly one fifth the price sensitivity of customers in the same segment (\( b_2 \)). Second, the cycle length is not monotone in any of the parameters, including \( \gamma \) and \( \theta \) (see also the discussion following Proposition 3); the values reported in Table 1 confirm the highly irregular behavior of the cycle length. Third, the potential loss when a firm ignores consumer heterogeneity can be substantial. If the firm assumes consumers are homogeneous in their perceived gains and losses (i.e., the market is not segmented), then the firm will charge a long-run uniform price shown in Table 2 (see Popescu and Wu, 2007, Theorem 4). When customers differ significantly in the strength of their reference effects (\( \gamma \geq 1.4 \)), the potential revenue loss can be as much as 46%, as shown by the bottom panel of Table 2.
<table>
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<td>7</td>
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<td>308</td>
<td>277</td>
<td>254</td>
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</table>

**Table 1:** The cycle length, price range and total revenue of the optimal dynamic pricing policy. A cycle length of 1 implies that $p^*(r_t)$ and $r_t$ converge to a steady state. The price range is evaluated for a cycle while ignoring the initial burn-in period.

4.2. Does Cyclic Pricing Occur for Empirically Observed Parameter Values?

The parameters we use below are based on two papers: Dahana and Terui (2006) and Moon et al. (2006). Dahana and Terui (2006) use the scanner panel data from two categories in the Japanese market, curry roux and instant coffee. The analysis in Moon et al. (2006) is based on the toilet tissue purchase records of 341 households over 114 weeks.

We consider two segments of customers in a market. Segment one is not subject to reference
Table 2: The optimal uniform pricing policy and its percentage loss compared to the optimal dynamic pricing policy reported in Table 1.

<table>
<thead>
<tr>
<th>( \gamma )</th>
<th>( \theta = 0.1 )</th>
<th>( \theta = 0.3 )</th>
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<td>1.65</td>
<td>1.64</td>
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The firm’s optimal pricing policy is illustrated in Figure 2. It is optimal for the firm to vary price over time and profit from the heterogeneity in reference effects of the two segments. The optimal total discounted revenue is 65.3. On the other hand, if the firm ignores the heterogeneity, then the optimal uniform price is \( p_t \equiv 0.42 \) which generates a total discounted revenue of 62.9.
4.3. Is Cyclic Pricing Optimal When Both Segments Are Loss-averse?

Consider a more general model in which both segments are loss-averse. Because the revenue functions are not concave, deriving the optimal dynamic pricing policy is an analytically intractable problem. Therefore, we rely on numerical experiments to showcase that our insights apply more generally.

Consider $a_1 = 1$, $b_1 = 0.3$, $a_2 = 4$, $b_2 = 2$, $\beta = 0.99$ and $\theta = 0.7$. In Figure 2 panel (a) is the base case in which we set $\gamma_1 = 0$, $\lambda_1 = 0.1$, $\gamma_2 = 1.5$, and $\lambda_2 = 2$. Because the reference effect in segment 2 is not sufficiently strong, the optimal dynamic price converges to a steady state. As segment-1 customers become less sensitive to losses ($\lambda_1$ decreases) or as segment-2 customers become more sensitive to gains ($\gamma_2$ increases), cyclic behavior eventually emerges. In this example, cyclic pricing dominates uniform pricing (w.r.t. revenues) when $\gamma_2 \geq 1.75$ or $\lambda_1 \leq 0.05$ (see, respectively, panels (b) and (c) of the figure). Yet when segment-2 customers become less sensitive to losses or segment-1 customers become more sensitive to gains, cyclic pricing does not emerge. To illustrate, we reduce $\lambda_2$ from 2 to 1.5 (note that $\lambda_2 \geq \gamma_2$ because of loss aversion) in panel (e) and increase $\gamma_1$ from 0 to 0.1 (loss aversion means that we must have $\lambda_1 \geq \gamma_1$) in panel (d). As these last two panels show, cyclic pricing is inferior to uniform pricing in such cases.

The experiment suggests that the cyclic behavior depends mostly on $\lambda_1$ and $\gamma_2$. In particular, a sufficiently large $\gamma_2$ and a sufficiently small $\lambda_1$ together give rise to cyclic pricing, whereas the values of $\lambda_2$ and $\gamma_1$ are not critical for the emergence of cyclic pricing. This is consistent with previous discussions. While the loss effect from segment 2 is limited so that $\lambda_2$ is not critical, it is the opposite for segment 1. Because the firm does not price out the high-valuation segment 1, the magnitude of $\lambda_1$ significantly restricts the profitability of cyclic pricing. From the numerical
Figure 3: The path of the optimal dynamic pricing policy and the reference price. Increasing $\gamma_2$ or decreasing $\lambda_1$ results in cyclic pricing. Decreasing $\lambda_2$ or increasing $\gamma_1$ does not change the pricing behavior.
examples, segment-2 customers have to be more sensitive to gains than segment-1 customers are to losses for cyclic pricing to be optimal.

5. Conclusion

The prevalent practice of offering discounts in consumer markets appears to be at odds with findings in the economic psychology literature that consumers are loss averse and dislike price variations. Those findings reflect the erosion, by discounts, of the internal reference price for a product—after which a return to the product’s “normal” price is perceived as a loss by consumers, reducing their demand for it. Hence retailers have been advised to keep their prices constant in the long term, a strategy that limits the role prices can play in managing consumer demand. However, recent developments in the OM literature offer plausible setups in which loss aversion does not preclude price variation. We add to this literature and suggest that the optimality of constant (long run) prices hinges on the assumption that consumers are homogenous in their gain/loss perceptions (i.e., that their response to perceived gains is similar to perceived losses of the same amount). Hence we model a market consisting of two segments of loss-averse consumers with heterogeneous gain/loss perceptions while making the (realistic) assumption that a firm cannot price-discriminate between the two segments. We also restore a key role of prices in managing demand from multiple segments and let the firm use price to turn off the demand from a consumer segment. We explain why the firm’s optimal dynamic pricing policy may be cyclic. One example is a markdown policy in which the firm charges a regular high price but occasionally offers a discount, thereby significantly boosting demand by appealing to consumers who are relatively more sensitive to gains.

Our model applies to fast moving consumer goods where repeated purchase experiences allow customers to anchor on the product’s price history to make purchase decisions. The choice of key parameters in Section 4 reflects this category of products for which the firm’s optimal policy is to vary prices over time and profit from the heterogeneity in reference effects across customer segments. Our results suggest that demand estimation procedures should account for differences in behavioral responses to a firm’s prices. Attempts to fit a single gain/loss parameter to market demand data will result in adopting suboptimal pricing policies (e.g., uniform instead of cyclic policies) and profit loss.

Our paper complements a nascent literature that emphasizes the importance of consumer heterogeneity for a firm’s pricing policies. Although our focus here is on heterogeneity in consumers’ gain/loss perceptions, consumers are likely to be heterogeneous also in their internal (or external) reference prices, shopping habits (see e.g. Wang 2016), and search behavior. Competitive markets offer another interesting context in which to study such characteristics. A natural extension to our model is to incorporate forward-looking behavior as consumers often wait in anticipation of discounts and promotions. Combining a memory-based reference price and forward-looking behavior
may be challenging however. We leave these ideas for future research.

References


A. Additional Results

A.1. Periodic Markdown

In this appendix, we provide an analytical characterization of the periodic markdown policy which we discussed in Section 2.1. Assume w.l.o.g. that the firm offers a discount at the end of the cycle.

Let $p_m(r_0) = p_m(r_1) = \cdots = p_m(r_{n-2}) = \frac{a_1}{2b_1}$ (the firm optimizes its revenue from segment 1) and $p_m(r_{n-1}) = \frac{a_1 + a_2 + \gamma r_{n-1}}{2(b_1 + b_2 + \gamma)}$ (the firm optimizes its revenue from both segments). By the evolution of the reference price, we have:

\[
\begin{align*}
r_1 &= \theta r_0 + (1-\theta) \frac{a_1}{2b_1}, \\
&\vdots \\
r_{n-1} &= \theta r_{n-1} + (1-\theta) \frac{a_1}{2b_1} = \theta^{n-1} r_0 + (1-\theta^{n-1}) \frac{a_1}{2b_1}, \\
r_n &= \theta r_{n-1} + (1-\theta) \frac{a_1 + a_2 + \gamma r_{n-1}}{2(b_1 + b_2 + \gamma)}.
\end{align*}
\]

Because $r_n = r_0$, we can use these equations to solve $r_0, r_1, \ldots, r_{n-1}$. In order for a periodic markdown pricing policy to be an optimal myopic policy, we require $r_i \leq \bar{r}^{\gamma}$ for $i = 0, \ldots, n-2$ so that $p_m(r_i) = \frac{a_1}{2b_1}$, and $r_{n-1} > \bar{r}^{\gamma}$ so that $p_m(r_{n-1}) = \frac{a_1 + a_2 + \gamma r_{n-1}}{2(b_1 + b_2 + \gamma)}$ (see case (iii) of Proposition 1).

These conditions lead to our next proposition.

**Proposition 6.** Suppose $\bar{r}^{\gamma} \in (r_0, a_1/2b_1)$. Then the myopic pricing policy is a periodic markdown policy with cycle length $n$ if and only if

\[
\theta^{n-2} r_0 + (1-\theta^{n-2}) \frac{a_1}{2b_1} < \bar{r}^{\gamma} < \theta^{n-1} r_0 + (1-\theta^{n-1}) \frac{a_1}{2b_1};
\]

Here $r_0 = \frac{b(1-\theta^{n-1})(a_1/2b_1) + (1-\theta)b}{1-\theta^{n-1}b}$ is the initial reference price, $b = \theta + (1-\theta) \frac{\gamma}{2(b_1 + b_2 + \gamma)}$, and $l = \frac{a_1 + a_2}{2(b_1 + b_2 + \gamma)}$.

The periodic markdown policy depends only on perceived gains and not on perceived losses—for the same reason discussed after Proposition 2. Moreover, the parameter values that give rise to cyclic pricing (case (iv) of Proposition 2) may not satisfy the condition in Proposition 6 for some $n$. Such circumstances may cause the emergence of cyclic behavior more complex than periodic markdowns.

As $\gamma$ increases, $\bar{r}^{\gamma}$ decreases and $r_{n-1} > \bar{r}^{\gamma}$ tends to be satisfied for smaller $n$. This implies that a stronger reference effect $\gamma$ may lead to more frequent discounts. The reason is that the firm finds it more profitable to price-skim segment 2 for a larger $\gamma$, which it should do more frequently even if the reference price is not very high. This approach does not run afoul of the bifurcation phenomenon (see text following Proposition 3) because we address only periodic markdown policies.
and ignore cyclic pricing policies of greater complexity.

To illustrate, let $a_1 = 1$, $b_1 = 0.1$, $a_2 = 1.5$, $b_2 = 2$, $\gamma = 0.7$, $\lambda = 0.75$, $\theta = 0.5$, and $r_0 = 2.71$. The cycle length is 3.

![Figure 4: Reference price path (blue dashed lines) and optimal myopic pricing path (red dashed lines) for $a_1 = 1$, $b_1 = 0.1$, $a_2 = 1.5$, $b_2 = 2$, $\gamma = 0.7$, $\lambda = 0.75$, $\theta = 0.5$, and $r_0 = 2.71$. The cycle length is 3.](image)

A.2. Peak-end Rule

The ‘peak-end rule’ is a recursive reference point formation mechanism which assumes consumers anchor on the most recent and the minimum prices paid for a product (Nasiry and Popescu, 2011), or $r_t = \theta m_{t-1} + (1 - \theta)p_{t-1}$ where $m_{t-1} = \min(m_{t-2}, p_{t-1})$.

Because the peak-end rule assigns a positive weight on the minimum price, it cannot be represented by deterministic weights. Nevertheless, we can still show that for sufficiently large $\gamma$, the firm’s optimal policy does not admit a steady state.

**Theorem 2.** Assume the reference point follows the peak-end rule. Then the optimal pricing policy $p_t^*$ for $t = 1, 2, \ldots$ does not admit a steady state for a sufficiently large $\gamma$. That is, there exists $\epsilon > 0$ such that for all $T \geq 1$, $\max_{t \geq T} p_t^* - \min_{t \geq T} p_t^* > \epsilon$.

B. Proofs

We first present the following lemma in order to prove Proposition [1].
Lemma 1. Define
\[ \pi^\gamma(p, r) = p((a_1 + a_2) - (b_1 + b_2)p + \gamma(r - p)) \quad \text{and} \quad \pi^\lambda(p, r) = p((a_1 + a_2) - (b_1 + b_2)p + \lambda(r - p)). \]

Assume the firm sells to both segments. Then
\[ p_a(r) \triangleq \arg\max_p \{ p((a_1 + a_2) - (b_1 + b_2)p + \gamma(r - p) + \gamma(r - p)) \} \]
\[ = \begin{cases} 
  p^\gamma(r) & \text{if } r \geq r^\gamma, \\
  r & \text{if } r^\lambda \leq r \leq r^\gamma, \\
  p^\lambda(r) & \text{if } r \leq r^\lambda; 
\end{cases} \quad (4) \]

Proof of Lemma 1. Consider the objective function
\[ f(p, r) \triangleq -p((a_1 + a_2) - (b_1 + b_2)p + \min\{\gamma(r - p), \lambda(r - p)\}). \]

The original problem is equivalent to finding the global minimizer of \( f(p, r) \) when given \( r \geq 0 \). It is easy to verify that \( f(p, r) \) is convex in \( p \). Because \( f \) is not differentiable at \( p = r \), we consider its subdifferential \( \partial_p f(p, r) \):
\[ \partial_p f(p, r) = \{ c \mid f(p + \Delta p, r) - f(p, r) \geq c \Delta p \ \forall \Delta p \}. \]

In other words, \( \partial_p f(p, r) \) is the set of the slopes of all lines that (a) pass through \( (p, f(p, r)) \) and (b) lie below the graph of \( f(p, r) \). For differentiable points, \( \partial_p f(p, r) \) is simply its derivative. Observe that
\[ \partial_p f(p, r) = \begin{cases} 
  (a_1 + a_2 + \gamma r) - 2(b_1 + b_2 + \gamma)p & \text{if } p > r, \\
  (a_1 + a_2 + \lambda r) - 2(b_1 + b_2 + \lambda)p & \text{if } p < r, \\
  \{(a_1 + a_2 + \xi r) - 2(b_1 + b_2 + \xi)p \mid \xi \in [\gamma, \lambda]\} & \text{if } p = r. 
\end{cases} \]

By the definition of a subdifferential, \( p_a(r) \) is a global minimizer of \( f(p, r) \) if and only if \( 0 \in \partial_p f(p_a(r), r) \). The equivalent formal expression is
\[ p_a(r) = \begin{cases} 
  \frac{a_1 + a_2 + \gamma r}{2(b_1 + b_2 + \gamma)} & \text{if } \frac{a_1 + a_2 + \gamma r}{2(b_1 + b_2 + \gamma)} > r, \\
  \frac{a_1 + a_2 + \lambda r}{2(b_1 + b_2 + \lambda)} & \text{if } \frac{a_1 + a_2 + \lambda r}{2(b_1 + b_2 + \lambda)} < r, \\
  r & \text{if } (a_1 + a_2 + \xi r) - 2(b_1 + b_2 + \xi)r = 0 \text{ for some } \xi \in [\gamma, \lambda]; 
\end{cases} \]
Lemma 2. Assume that Assumption 2 holds and that segment-2 customers are loss neutral (i.e., \( \gamma = \lambda \)). Then

\[
p_a(r) = \begin{cases} 
p^\gamma(r) & \text{if } r > r^\gamma, \\
p^\lambda(r) & \text{if } r < r^\lambda, \\
r & \text{if } r \in [r^\lambda, r^\gamma].
\end{cases}
\]

This completes the proof.

Proof of Proposition 1. To prove this proposition, we first establish the following lemma.

**Lemma 2.** Assume that Assumption 2 holds and that segment-2 customers are loss neutral (i.e., \( \gamma = \lambda \)). Then

\[
\argmax_{p \in [0, a_1/b_1]} \{ \Pi(p, r) \} = \begin{cases} 
a_1/2b_1 & \text{if } 0 \leq r < \min \{ \bar{r}, \frac{a_1}{b_1} \}, \\
\frac{a_1 + a_2 + \gamma r}{2(b_1 + b_2 + \gamma)} & \text{if } \max \{ \bar{r}, 0 \} \leq r \leq \frac{a_1}{b_1};
\end{cases}
\]

in this expression, \( \bar{r} = \frac{a_1}{\gamma} \sqrt{\frac{b_1 + b_2 + \gamma}{b_1}} - \frac{a_1 + a_2}{\gamma} \).

**Proof.** For given \( p \) and for \( r \in [0, a_1/b_1] \), we have

\[
\Pi(p, r) = pD(p, r) = \begin{cases} 
-(b_1 + b_2 + \gamma)p^2 + (a_1 + a_2 + \gamma r)p & \text{if } 0 \leq p \leq \frac{a_2 + \gamma r}{b_2 + \gamma}, \\
-b_1p^2 + a_1p & \text{if } \frac{a_2 + \gamma r}{b_2 + \gamma} < p \leq \frac{a_1}{b_1}.
\end{cases}
\]

It is obvious that \( \argmax_p\{p(a_1 - b_1p)\} = a_1/2b_1 \) and that \( \max_p\{p(a_1 - b_1p)\} = a_1^2/4b_1 \). Furthermore,

\[
\argmax_p\{p(a_1 + a_2 - (b_1 + b_2)p + \gamma(r - p))\} = \frac{a_1 + a_2 + \gamma r}{2(b_1 + b_2 + \gamma)}
\]

and so

\[
\max_p\{p(a_1 + a_2 - (b_1 + b_2)p + \gamma(r - p))\} = \frac{(a_1 + a_2 + \gamma r)^2}{4(b_1 + b_2 + \gamma)}.
\]

The two local maxima are equal for all values of \( r \) that solve

\[
\frac{a_1}{2b_1} = \frac{a_1 + a_2 + \gamma r}{2\sqrt{b_1 + b_2 + \gamma}},
\]

which admits two solutions. The first solution, \( r_s \), satisfies

\[
\frac{a_1}{2\sqrt{b_1}} = \frac{a_1 + a_2 + \gamma r_s}{2\sqrt{b_1 + b_2 + \gamma}};
\]

the second solution, \( \bar{r} \), satisfies

\[
\frac{a_1}{2\sqrt{b_1}} = \frac{a_1 + a_2 + \gamma \bar{r}}{2\sqrt{b_1 + b_2 + \gamma}}.
\]

Observe that \( r_s = \frac{a_1}{\gamma} \sqrt{\frac{b_1 + b_2 + \gamma}{b_1}} - \frac{a_1 + a_2}{\gamma} \) is negative and less than \( \bar{r} \). By the properties of quadratic functions, if \( r > \max\{\bar{r}, r_s\} \) then

\[
\frac{a_1^2}{4b_1} < \frac{(a_1 + a_2 + \gamma r)^2}{4(b_1 + b_2 + \gamma)}
\]

and if \( r \) is between \( \bar{r} \) and \( r_s \) then

\[
\frac{a_1^2}{4b_1} > \frac{(a_1 + a_2 + \gamma r)^2}{4(b_1 + b_2 + \gamma)}.
\]
\[
\frac{a_1^2}{4b_1} > \frac{(a_1 + a_2 + \gamma r)^2}{(b_1 + b_2 + \gamma)^2}.
\]
Therefore,
\[
\text{argmax}_p \{ \Pi(p, r) \} = \begin{cases} 
\frac{a_1}{2b_1} & \text{if } \frac{a_1^2}{4b_1} > \frac{(a_1 + a_2 + \gamma r)^2}{(b_1 + b_2 + \gamma)^2}, \\
\frac{a_1 + a_2 + \gamma r}{2(b_1 + b_2 + \gamma)} & \text{if } \frac{a_1^2}{4b_1} \leq \frac{(a_1 + a_2 + \gamma r)^2}{(b_1 + b_2 + \gamma)^2}.
\end{cases}
\]

This completes the proof of Lemma 2.

We now proceed with the proof of Proposition 1. Our first task is to show that \(\bar{r}^\lambda\), \(\bar{r}^\gamma\), and \(\bar{r}^0\) each satisfy the following statements. For \(r \geq 0\),

\[
p^\lambda(r)(a_1 + a_2 - (b_1 + b_2)p^\lambda(r) + \lambda(r - p^\lambda(r))) < \Pi_1 \left( \frac{a_1}{2b_1} \right) \iff r < \bar{r}^\lambda, \tag{5}
\]

\[
p^\gamma(r)(a_1 + a_2 - (b_1 + b_2)p^\gamma(r) + \gamma(r - p^\gamma(r))) < \Pi_1 \left( \frac{a_1}{2b_1} \right) \iff r < \bar{r}^\gamma; \tag{6}
\]

\[
r(a_1 + a_2 - (b_1 + b_2)r) \begin{cases} < \Pi_1 \left( \frac{a_1}{2b_1} \right) & \text{if } r < \bar{r}^0, \\
> \Pi_1 \left( \frac{a_1}{2b_1} \right) & \text{if } r \in (\bar{r}^0, r^\gamma). \tag{7}
\end{cases}
\]

Here we prove only (7) because (5) and (6) follow directly from Lemma 2. Note that \(r(a_1 + a_2 - (b_1 + b_2)r) = \Pi_1 \left( \frac{a_1}{2b_1} \right)\) has two solutions: \(r^0\), which is less than \(\frac{a_1 + a_2}{2(b_1 + b_2)}\); and \(r_s\), which is greater than that fraction. Equation (7) then follows because \(r^\gamma < \frac{a_1 + a_2}{2(b_1 + b_2)}\).

We can now show case 1 of the proposition. Since \(\bar{r}^0 \leq r^\lambda\), it follows from (7) that \(\Pi(r^\lambda, r^\lambda) \geq \Pi_1(a_1/2b_1)\). And because \(r^\lambda = p^\lambda(r^\lambda)\), we have \(\Pi(p^\lambda(r^\lambda), r^\lambda) \geq \Pi_1(a_1/2b_1)\). This inequality implies that, by (5), \(\bar{r}^\lambda \leq r^\lambda\). At this point, optimizing \(\Pi(p, r)\) requires only that we compare the two local maxima: one maximum at price \(p_a(r)\), which maximizes revenue from the aggregated market; and one maximum at \(a_1/2b_1\), which maximizes only segment-1 revenue.

If \(r \leq \bar{r}^\lambda \leq r^\lambda\) then, by (4), \(p_a(r) = \frac{a_1 + a_2 + \lambda r}{2(b_1 + b_2 + \lambda)}\) in the aggregated market. By Lemma 2,

\[
\Pi_1 \left( \frac{a_1}{2b_1} \right) \geq \Pi \left( \frac{a_1 + a_2 + \lambda r}{2(b_1 + b_2 + \lambda)}, r \right). \tag{8}
\]

Similarly, if \(r \in (\bar{r}^\lambda, r^\lambda)\) then, by Lemma 2,

\[
\Pi_1 \left( \frac{a_1}{2b_1} \right) < \Pi \left( \frac{a_1 + a_2 + \lambda r}{2(b_1 + b_2 + \lambda)}, r \right). \tag{9}
\]

As a consequence, \(p_m(r) = \frac{a_1 + a_2 + \lambda r}{2(b_1 + b_2 + \lambda)}\). For \(r \in [r^\lambda, a_1/b_1]\), we have

\[
\Pi(p_m(r), r) \geq \Pi \left( \frac{a_1 + a_2 + \lambda r}{2(b_1 + b_2 + \lambda)}, r \right) \geq \Pi_1 \left( \frac{a_1}{2b_1} \right). \tag{10}
\]

It follows that \(p_m(r) = p_a(r)\) for \(r \in [\bar{r}^\lambda, a_1/b_1]\).

For case 2 in Proposition 1 by the definition of \(r^0\) we have \(\Pi(p_a(r^0), r^0) = \Pi(r^0, r^0) = \Pi_1(a_1/2b_1)\). Then \(\Pi(p_a(r), r) < \Pi_1(a_1/2b_1)\) for \(r < r^0\) because \(\Pi(p, r)\) is an increasing function of \(r\) for any given \(p\). We therefore have that \(p_m(r) = a_1/2b_1\) for \(r \in [0, \bar{r}^0]\) and that \(p_m(r) = p_a(r)\)
for \( r \in [\bar{r}, a_1/b_1] \).

The proof for case 3 is similar to that for case 1 and so has been omitted.

**Proof of Proposition 2.** Because \( p_a(r) \leq p_a\left(\frac{a_1}{b_1}\right) = p^\gamma\left(\frac{a_1}{b_1}\right) = \frac{a_1+\alpha_2+\gamma(a_1/b_1)}{2(b_1+\alpha_2+\gamma)} < \frac{a_1}{2b_1} \), there is a downward jump at the discontinuity of \( p_m(r) \) in all three cases of Proposition 1.

In case 1 of that proposition, note that \( \bar{r}^\lambda \leq r^\lambda = \frac{a_1+\alpha_2}{\lambda+2(b_1+\alpha_2+\gamma)} < \frac{a_1}{2b_1} \). Hence the equation \( p_m(r) = \frac{a_1}{2b_1} = r \) has no solutions for \( r < \bar{r}^\lambda \). For \( r \geq \bar{r}^\lambda \), the equation \( p_m(r) = p_a(r) = r \) has solutions \( r \in [r^\lambda, r^\gamma] \); this scenario corresponds to case (i) in Proposition 2. Similarly, for case 2 in Proposition 1 any \( r \in [\bar{r}^\lambda, r^\gamma] \) is a solution to \( p_m(r) = r \); this scenario corresponds to case (ii) in Proposition 2.

For case 3 in Proposition 1 there are two possibilities. If \( \bar{r}^\gamma < a_1/2b_1 \), then \( p_m(r) = r \) has one solution: \( a_1/2b_1 \). Otherwise, \( p_m(r) = r \) has no solutions. Those two possibilities are illustrated by panels (c) and (d), respectively, in Figure 1 which correspond to the respective cases (iii) and (iv) of Proposition 2.

**Proof of Proposition 3.** For cases (i), (ii), and (iii) of Proposition 2 there exist steady states. In all three cases, \( p_m(\cdot) \) consists of piecewise linear functions and has at most one discontinuity, and there is no linear segment whose slope exceeds 1. We use case (i) in Proposition 2 to show that \( r_t \) and \( p_m(r_t) \) converge to one steady state; the claims for cases (ii) and (iii) follow similarly. When \( r_t < \bar{r}^\lambda \), we must have \( r_{t+s} \geq \bar{r}^\lambda \) for some \( s > 0 \) because \( r_{t+1} - r_t = (1-\theta)(p_m(r_t) - r_t) \geq (1-\theta)(\lim_{r \rightarrow \bar{r}^\lambda} p_m(r) - \bar{r}^\lambda) > 0 \) is bounded away from zero.

When \( r^\lambda > r_t \geq \bar{r}^\lambda \), we can show that \( |r_{t+1} - r^\lambda| = \theta |r_t - r^\lambda| + (1-\theta) |p_m(r_t) - r^\lambda| = \left(\theta + (1-\theta) \frac{\lambda}{2(b_1+\alpha_2+\gamma)}\right) |r_t - r^\lambda| \) is a contraction. Therefore, \( r_t \) converges to \( r^\lambda \) as does \( p_m(r_t) \).

When \( r^\lambda < r_t \leq r^\gamma \), the reference point \( r_t \) is already a steady state. When \( r_t > r^\gamma \), we show that \( r_t \) converges to \( r^\gamma \) much as when \( r^\lambda > r_t \geq \bar{r}^\lambda \). As a result, \( r_t \) and \( p_m(r_t) \) always converge to a steady state. The convergence for case (i) is illustrated in panel (a) of Figure 1. Cases (ii) and (iii) of Proposition 2 can be proved similarly and are presented graphically in, respectively, panels (b) and (d) of Figure 1.

Finally, for case (iv) in Proposition 2 there is no steady state. The trajectory of \( r_t \) converges to a stable cycle, and so does that of \( p_m(r_t) \) (see Gardini and Tramontana 2010, Sec. 5.1). This completes the proof.

Before proving Proposition 4 we show the following lemma.

**Lemma 3.** For any \( r \in [0, a_1/b_1] \) and \( 0 < \Delta r < a_1/b_1 - r \), we have \( 0 \leq \frac{V(r+\Delta r) - V(r)}{\Delta r} \leq \frac{\lambda a_1}{(1-\beta) b_1} \).
Proof of Lemma 3. Define the value iteration

\[ V^{(0)} \equiv 0, \]
\[ V^{(k+1)}(r) = \max_p \{ \Pi(p, r) + \beta V^{(k)}(\theta r + (1 - \theta)p) \}, \quad k = 1, 2, \ldots. \]

Clearly, \( V(r) = \lim_{k \to \infty} V^{(k)}(r) \). We will prove the lemma by induction. We have \( V^{(0)}(r + \Delta r) - V^{(0)}(r) = 0 \) for \( k = 0 \); for \( k \geq 0 \),

\[ V^{(k+1)}(r + \Delta r) - V^{(k+1)}(r) = \max_p \{ \Pi(p, r + \Delta r) + \beta V^{(k)}(\theta (r + \Delta r) + (1 - \theta)p) \} \]
\[ - \max_p \{ \Pi(p, r) + \beta V^{(k)}(\theta r + (1 - \theta)p) \}. \]

Because \( \Pi(p, r) \) and \( V^{(k)}(\theta r + (1 - \theta)p) \) are both increasing in \( r \), it must be that \( V^{(k+1)}(r + \Delta r) - V^{(k+1)}(r) \geq 0 \). We need to show that \( V^{(k+1)}(r + \Delta r) - V^{(k+1)}(r) \leq \frac{\gamma a_1}{(1 - \beta \theta) b_1} \Delta r \). Toward that end, we write

\[ \max_p \{ \Pi(p, r + \Delta r) + \beta V^{(k)}(\theta (r + \Delta r) + (1 - \theta)p) \} - \max_p \{ \Pi(p, r) + \beta V^{(k)}(\theta r + (1 - \theta)p) \} \]
\[ \leq \max_p \{ \Pi(p, r + \Delta r) - \Pi(p, r) + \beta (V^{(k)}(\theta (r + \Delta r) + (1 - \theta)p) - V^{(k)}(\theta r + (1 - \theta)p)) \}. \]

The function \( \Pi(p, r) \) has one of the following three forms: \( p((a_1 + a_2) - (b_1 + b_2)p + \gamma(r - p)), \)
\( p((a_1 + a_2) - (b_1 + b_2)p + \lambda r - p)), \) or \( p(a_1 - b_1 p) \). Therefore, \( \Pi(p, r + \Delta r) - \Pi(p, r) \leq \lambda p \Delta r \leq \lambda a_1 \Delta r / b_1 \).

It now follows from the inductive hypothesis that

\[ V^{(k+1)}(r + \Delta r) - V^{(k+1)}(r) \leq \max_p \left\{ \frac{a_1 \gamma}{b_1} \Delta r + \beta \frac{\gamma a_1}{(1 - \beta \theta) b_1} \theta \Delta r \right\} \leq \frac{\gamma a_1}{(1 - \beta \theta) b_1} \Delta r. \]

Hence the inequality \( 0 \leq V^{(k)}(r + \Delta r) - V^{(k)}(r) \leq \frac{\gamma a_1}{(1 - \beta \theta) b_1} \Delta r \) holds for all \( k \geq 0 \) as well as for \( V(r) \). This completes the proof.

Proof of Proposition 3. Similarly to the proof of Lemma 3, this proof uses the value iterations

\[ V^{(0)}(r) = V^{(0)}(r) \equiv 0, \]
\[ V^{(k+1)}(r) = \Pi(p_m(r), r) + \beta V^{(k)}_m(\theta r + (1 - \theta)p_m(r)), \]
\[ V^{(k+1)}(r) = \Pi(p^*(r), r) + \beta V^{(k)}(\theta r + (1 - \theta)p^*(r)); \]

thus \( V^{(\infty)}(r) = V(r) \) and \( V^{(\infty)}(r) = V_m(r) \). Next we show (by induction) that \( V^{(k)}(r) - V^{(k)}_m(r) \leq \frac{\gamma a_1}{(1 - \beta \theta) b_1} \Delta r \) holds for all \( k \geq 0 \) as well as for \( V(r) \).
\[ \frac{\beta(1-\theta)\lambda a_1^2}{(1-\beta)(1-\beta)\theta b_1^2}. \] For \( k = 0 \), we have \( V_m \equiv V \) and so the inequality holds. For \( k \geq 0 \),

\[
V^{(k+1)}(r) - V_m^{(k+1)}(r) = \Pi(p^*(r), r) - \Pi(p_m(r), r) + \beta(V^{(k)}(\theta r + (1 - \theta)p^*(r)) - V_m^{(k)}(\theta r + (1 - \theta)p_m(r)))
\leq \beta(V^{(k)}(\theta r + (1 - \theta)p^*(r)) - V_m^{(k)}(\theta r + (1 - \theta)p_m(r)))
= \beta(V^{(k)}(\theta r + (1 - \theta)p^*(r)) - V^{(k)}(\theta r + (1 - \theta)p_m(r)))
+ \beta(V^{(k)}(\theta r + (1 - \theta)p_m(r)) - V_m^{(k)}(\theta r + (1 - \theta)p_m(r)));
\]

(9)

here the inequality follows because \( p_m(r) \) maximizes \( \Pi(p, r) \). By Lemma 3, the first term of (9) is bounded by

\[
(p^*(r) - p_m(r)) \frac{\beta(1-\theta)\lambda a_1}{(1-\beta)b_1} \leq \frac{\beta(1-\theta)\lambda a_1^2}{(1-\beta)\theta b_1^2}.
\]

According to the induction hypothesis, the second term of (9) is bounded by \( \frac{\beta^2(1-\theta)\lambda a_1^2}{(1-\beta)(1-\beta)\theta b_1^2} \). Therefore,

\[
V^{(k+1)}(r) - V_m^{(k+1)}(r) \leq \frac{\beta(1-\theta)\lambda a_1^2}{(1-\beta)b_1^2} + \frac{\beta^2(1-\theta)\lambda a_1^2}{(1-\beta)(1-\beta)\theta b_1^2} = \frac{\beta(1-\theta)\lambda a_1^2}{(1-\beta)(1-\beta)\theta b_1^2}.
\]

Thus the inequality holds for all \( k \geq 0 \). Taking \( k \to \infty \) now proves the result for \( V_m(r) - V(r) \).

**Proof of Proposition 5.** We will prove the proposition by contradiction. Suppose a steady state \( p_s \) exists. Then, by equation (2), \( V(p_s) = \Pi(p_s, p_s) + \beta V(p_s) \) or (equivalently) \( V(p_s) = \frac{\Pi(p_s, p_s)}{1-\beta} \). We will show that if \( \gamma \) is sufficiently large then setting price \( p_s \) is not optimal when \( r = p_s \); that is, we can find another pricing policy that generates a higher discounted revenue than \( \frac{\Pi(p_s, p_s)}{1-\beta} \).

For this purpose, we start by showing that \( V(p_s) \) (or \( \Pi(p_s, p_s) \)) has both an upper and a lower bound that are positive and independent of \( \gamma \) and \( \lambda \). The lower bound is given by \( \Pi_1(a_2/2b_1) \), which is the optimal single-period revenue from segment 1. That revenue is clearly less than the total revenue of the optimal dynamic policy \( V(p_s) \) and is also independent of \( \gamma \) and \( \lambda \). The upper bound of \( \Pi(p_s, p_s) \) is given by \( \Pi(p_s, p_s) = \max\{p_s(a_1-b_1p_s), p_s(a_1+a_2-(b_1+b_2)p_s)\} \leq \max\{a_2^2, (a_1+a_2)^2\} \}

which is also independent of \( \gamma \) and \( \lambda \).

Define a continuous function \( f(p) \equiv \Pi(p,p) \). Clearly, \( f(p) \) is independent of both \( \gamma \) and \( \lambda \); furthermore, \( f(0) = 0 \). Since \( f(p_s) = (1-\beta)\Pi(p_s) \) is bounded away from zero, it follows (by the continuity of \( f(p) \) at \( p = 0 \)) that \( p_s \) is outside a neighborhood \((0, \delta)\) of zero for some \( \delta > 0 \). Moreover, the choice of \( \delta \) is independent of \( \gamma \) and \( \lambda \).

Next consider the following pricing policy for \( r_0 = p_s \). Let \( p_k = \delta/2 \) for \( k \geq 0 \)—that is, charge a constant price \( \delta/2 \). Then the revenue generated at \( t = 0 \) is

\[
\Pi\left(\frac{\delta}{2}, p_s\right) \geq \frac{\delta}{2}\left(a_2 - b_2\frac{\delta}{2} + \gamma\left(p_s - \frac{\delta}{2}\right)\right) \geq \frac{\gamma\delta^2}{4} + \frac{\delta}{2}\left(a_2 - b_2\frac{\delta}{2}\right).
\]

As \( \gamma \) increases, the RHS of this inequality can grow without bound. In particular, for a sufficiently
large $\gamma$ we have $\Pi(\delta/2, p_s) > \Pi(p_s, p_s)/(1 - \beta) = V(p_s)$ because $V(p_s)$ has an upper bound that is independent of $\gamma$ and $\lambda$. This outcome contradicts the optimality of $V(p_s)$, thereby completing the proof.

\textbf{Proof of Theorem 1.} Define a constant $C_1 = \frac{1}{1 - \beta} \left( \frac{a_1^2}{2b_1} + \frac{a_2^2}{4b_2} \right)$. Our goal is to show that for $\gamma > \max \left\{ \frac{64C_1b_1^2}{a_1^2}, \frac{16\beta - K}{a_1^2}, \frac{16\beta - K}{a_1^2}, \frac{4\beta - K}{a_1^2}, \frac{16\beta - K}{a_1^2} \right\}$, the optimal policy does not admit a steady state. We prove the claim by contradiction. Suppose the optimal price has a steady state, i.e., for all $\epsilon > 0$, we can find a $T_1(\epsilon) > 0$ such that $\max_{t \geq T_1} p_t^* - \min_{t \geq T_1} p_t^* < \epsilon$. Consider a particular $\epsilon = \min \left\{ \frac{a_1}{b_1}, \frac{a_2}{b_2} \right\}$ and such $T_1$. Without loss of generality, for $t \geq T_1$, we can find $\bar{p}$ such that $|p_t^* - \bar{p}| < \epsilon/2$. Because the reference formation process is asymptotically oblivious, for any small number $\epsilon_1 > 0$, we can find $T_2 > T_1$ such that for $t \geq T_2$, we have $\sum_{i=0}^{T_1} w_{t,i} < \epsilon_1$. In particular, we can choose $\epsilon_1$ sufficiently small so that

$$\begin{align*}
r_t^* &= w_{t-1}^* r_t^* + \sum_{i=0}^{t} w_{t,i} p_t^* \geq \sum_{i=T_1+1}^{t} w_{t,i} p_t^* \geq (\bar{p} - \epsilon/2)(1 - \epsilon_1) \geq \bar{p} - \epsilon \
r_t^* &\leq \epsilon_1 a_1/b_1 + \sum_{i=T_1+1}^{t} w_{t,i} p_t^* \leq \bar{p} + \epsilon/2 + \epsilon_1 a_1/b_1 \leq \bar{p} + \epsilon,
\end{align*}$$

where $r_t^*$ is the reference price associated with the policy $p_t^*$. Therefore, for $t \geq T_2$, we have $|p_t^* - \bar{p}| < \epsilon/2$ and $|r_t^* - \bar{p}| < \epsilon$.

Consider the revenues generated by the optimal policy from $T_2$ onward, discounted to period $T_2$: $\sum_{i=0}^{\infty} \beta^i \Pi(p_{T_2+i}^*, r_{T_2+i}^*)$. By the above results, we have

$$\begin{align*}
\sum_{i=0}^{\infty} \beta^i \Pi(p_{T_2+i}^*, r_{T_2+i}^*) &\leq \sum_{i=0}^{\infty} \beta^i \left( p_{T_2+i}^* D_1(p_{T_2+i}^*) + p_{T_2+i}^* D_2(p_{T_2+i}^*, r_{T_2+i}^*) \right) \\
&\leq \sum_{i=0}^{\infty} \beta^i \left( \frac{a_1^2}{4b_1} + \frac{a_2^2}{4b_2} + \frac{a_1}{b_1} \gamma(r_{T_2+i}^* - p_{T_2+i}^*) \right) \\
&\leq \frac{1}{1 - \beta} \left( \frac{a_1^2}{4b_1} + \frac{a_2^2}{4b_2} + \frac{3a_1 \gamma}{2b_1} \right) \\
&\leq \frac{1}{1 - \beta} \left( \frac{a_1^2}{2b_1} + \frac{a_2^2}{4b_2} \right) = C_1
\end{align*}$$

where the last inequality holds because $\gamma c \leq a_1/6$.

We next show that $\bar{p} + \epsilon < a_1/2b_1$. Combined with the previous result that $|p_t^* - \bar{p}| < \epsilon/2$, this will imply that the steady state of the optimal policy is below $a_1/2b_1$. If this is not true, then we have $r_{T_2} \geq \bar{p} - \epsilon \geq a_1/2b_1 - 2\epsilon \geq a_1/4b_1$ (because $\epsilon \leq a_1/8b_1$). Consider the following policy $p_t'$: from $t = 0$ to $T_2 - 1$, $p_t' = \bar{p}_t'$; for $t = T_2$, $p_t' = \min\{a_1/8b_1, a_2/2b_2\}$; for $t \geq T_2 + 1$, $p_t' = 0$. Clearly, the two policies $p_t^*$ and $p_t'$ generate the same revenue before period $T_2$. Now consider the revenue

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generated in $T_2$ for the policy $p'_t$. Because $p'_{T_2} \leq r^*_{T_2} = r'_{T_2}$, we have
\[ \Pi(p'_{T_2}, r'_{T_2}) \geq p'_{T_2} \left( a_2 - b_2 p'_{T_2} + \gamma \left( \frac{a_1}{4b_1} - p'_{T_2} \right) \right)^+. \tag{10} \]

If $a_1/8b_1 \leq a_2/2b_2$, then $p'_{T_2} = a_1/8b_1$ and
\[ \text{RHS of } (10) \geq \frac{a_1}{8b_1} \left( \gamma \frac{a_1}{8b_1} \right) \geq \gamma \frac{a_1^2}{64b_1^2} > C_1. \]

If $a_1/8b_1 > a_2/2b_2$, then $p'_{T_2} = a_2/2b_2$ and
\[ \text{RHS of } (10) \geq \frac{a_2}{2b_2} \left( \gamma \frac{a_2}{2b_2} \right) \geq \gamma \frac{a_2^2}{4b_2^2} > C_1. \]

Either way, the revenue generated in a single period $T_2$ by the new policy $p'_t$ is higher than $C_1$, which is the total discounted revenue generated by the optimal policy from $T_2$ onward. This contradicts the fact that $p^*_t$ is optimal. Thus, we have proved that $\bar{p} + \epsilon < a_1/2b_1$.

We next show that $p^*_t$ cannot be optimal. Consider the following policy $p'_t$: from $t = 0$ to $T_2 - 1$, $p'_t = p^*_t$; for $t = T_2$ to $T_2 + K - 1$, $p'_t = a_1/b_1$; for $t = T_2 + K$, $p'_t = \min\{a_2/2b_2, \delta a_1/4b_1\}$; for $t \geq T_2 + K + 1$, $p'_t = 0$. Now we compute the revenue generated in period $T_2 + K$. Because $p^*_t \leq \bar{p} + \epsilon < a_1/2b_1$ for $t = T_2, \ldots, T_2 + K - 1$, we have
\[
r'_{T_2+K} = w_{T_2+K,-1}r_0 + \sum_{i=0}^{T_2-1} w_{T_2+K,i} p^*_i + \sum_{i=T_2}^{T_2+K-1} w_{T_2+K,i} \frac{a_1}{b_1}
\geq w_{T_2+K,-1}r_0 + \sum_{i=0}^{T_2-1} w_{T_2+K,i} p^*_i + \sum_{i=T_2}^{T_2+K-1} w_{T_2+K,i} \left( p^*_i + \frac{a_1}{2b_1} \right)
\geq r^*_{T_2+K} + \frac{a_1}{2b_1} \sum_{i=T_2}^{T_2+K-1} w_{T_2+K,i} \geq r^*_{T_2+K} + \frac{a_1 \delta}{2b_1}.
\]

The last inequality follows from the fact that the reference formation is $(K, \delta)$-retentive. Then the revenue in period $T_2 + K$ is
\[ \Pi(p'_{T_2+K}, r'_{T_2+K}) \geq p'_{T_2+K} \left( a_2 - b_2 p'_{T_2+K} + \gamma \left( \frac{a_1 \delta}{2b_1} - p'_{T_2+K} \right) \right). \tag{11} \]

If $a_2/2b_2 \leq \delta a_1/4b_1$, then $p'_{T_2+K} = a_2/2b_2$ and
\[ \text{RHS of } (11) \geq \gamma \frac{a_2^2}{4b_2^2} > \beta^{-K} C_1. \]
If \( a_2/2b_2 > \delta a_1/4b_1 \), then \( p_{T_2+K} = a_1\delta/4b_1 \) and

\[
\text{RHS of (11)} \geq \gamma \frac{\delta^2 a_1^2}{16b_1^2} > \beta^{-K} C_1.
\]

In either way, the revenue generated by \( p_t \) in period \( T_2 + K \), discounted to \( T_2 \), is higher than \( C_1 \), the total discounted revenue generated by the optimal policy from \( T_2 \) onward. Because \( p_t \) and \( p_t^* \) generate the same revenue before \( T_2 \), it contradicts the fact that \( p_t^* \) is the optimal policy. Therefore, we have proved that \( p_t^* \) does not admit a steady state.

**Proof of Proposition 4.** Periodic markdown is an instance of case 3 in Proposition 1 or of case (iv) in Proposition 2. So for \( p_m(r) \) as derived in Proposition 1 we may write

\[
r_1 = \theta r_0 + (1 - \theta) \frac{a_1}{2b_1},
\]

\[
\vdots
\]

\[
r_{n-1} = \theta r_{n-1} + (1 - \theta) \frac{a_1}{2b_1} = \theta^{n-1} r_0 + (1 - \theta^{n-1}) \frac{a_1}{2b_1},
\]

\[
r_n = \theta r_{n-1} + (1 - \theta) \frac{a_1 + a_2 + r_0}{2(b_1 + b_2 + \gamma)}
= \left( \theta + (1 - \theta) \frac{\gamma}{2(b_1 + b_2 + \gamma)} \right) \left( \theta^{n-1} r_0 + (1 - \theta^{n-1}) \frac{a_1}{2b_1} \right) + \frac{(1 - \theta)(a_1 + a_2)}{2(b_1 + b_2 + \gamma)} = r_0.
\]

From the last of these equalities it follows that \( r_0 = \frac{b(1-\theta^{n-1})(a_1/2b_1)+(1-\theta)l}{1-\theta^{n-1}b} \), where \( b = \theta + (1 - \theta) \frac{\gamma}{2(b_1 + b_2 + \gamma)} \) and \( l = \frac{a_1 + a_2}{2(b_1 + b_2 + \gamma)} \).

For periodic markdown to be a myopically optimal pricing policy, it must match \( p_m(r) \) in case 3 of Proposition 1. The implication is that \( r_{n-2} < \bar{r} \gamma \) and also \( r_{n-1} > \bar{r} \gamma \). Therefore,

\[
r_{n-2} < \bar{r} < r_{n-1} \iff \theta^{n-2} r_0 + (1 - \theta^{n-2}) \frac{a_1}{2b_1} < \bar{r} \gamma < \theta^{n-1} r_0 + (1 - \theta^{n-1}) \frac{a_1}{2b_1},
\]

where we have used the expression \( \bar{r}_i = \theta^i r_0 + (1 - \theta^i) \frac{a_1}{2b_1} \) for \( i \leq n-2 \). This completes the proof. ■

**Proof of Theorem 3.** Because the peak-end rule cannot be represented by \( 1 \), we cannot directly use the same method as in the proof of Theorem 1. The key difference is that, for a pricing policy that converges to a steady state, the reference price may converge to a different steady state. This does not turn out to be a problem as the peak-end rule implies that the steady state of the reference price is always lower than the steady state price. Therefore, we can use a similar perturbation approach used in Theorem 1.

Define a constant \( C_1 = \frac{1}{\beta(1-\theta)^2 a_1^2} \). We will show that for \( \gamma > \max \left\{ \frac{64C_1 b_1^2}{\beta(1-\theta)^2 a_1^2}, \frac{4C_1 b_2^2}{\beta a_1^2} \right\} \), the optimal policy does not admit a steady state.

We prove the claim by contradiction. Suppose the optimal price has a steady state, i.e., for
all $\epsilon > 0$, we can find a $T_1(\epsilon) > 0$ so that $\max_{t \geq T_1} p_t^* - \min_{t \geq T_1} p_t^* < \epsilon$. Consider a particular $\epsilon = \min \left\{ \frac{a_1}{8}, \frac{a_1}{8(1-\theta)b_1} \right\}$ and such $T_1$. Without loss of generality, for $t \geq T_1$, we can find $\bar{p}$ such that $|p_t^* - \bar{p}| < \epsilon/2$. Denote $\bar{p} = \min_{1 \leq t \leq T_1} \{p_t^*\}$. Clearly, $\bar{p} \leq p_{T_1}^* \leq \bar{p} + \epsilon/2$. Therefore, for $t \geq T_1 + 2$

$$r_t^* = (1 - \theta)p_{t-1}^* + \theta \min \left\{ \bar{p}, p_{T_1+1}^*, \ldots, p_{t-1}^* \right\} \leq (1 - \theta)\bar{p} + \theta \bar{p} + \frac{\epsilon}{2}$$

and

$$r_t^* \geq (1 - \theta)p_{t-1}^* + \theta \min \left\{ \bar{p}, \bar{p} - \frac{\epsilon}{2} \right\} \geq (1 - \theta)\bar{p} + \theta \bar{p} - \epsilon,$$

where $r_t^*$ is the reference prices associated with the policy $p_t^*$. Therefore, for $t \geq T_1 + 2$, we have $|p_t^* - \bar{p}| < \epsilon/2$ and $|r_t^* - (1 - \theta)\bar{p} - \theta \bar{p}| < \epsilon$. As a result, $r_t^* \leq (1 - \theta)\bar{p} + \theta \bar{p} + \epsilon \leq \bar{p} + 3\epsilon/2 \leq p_t^* + 2\epsilon$; and $r_t^* \geq (1 - \theta)\bar{p} + \theta \bar{p} - \epsilon \geq (1 - \theta)(p_t^* - \epsilon/2) - \epsilon \geq (1 - \theta)p_t^* - 2\epsilon$.

Consider the revenues generated by the optimal policy from $T_1 + 2$ onward, discounted to period $T_1 + 2$: $\sum_{i=0}^{\infty} \beta^i \Pi(p_{T_1+2+i}^*, r_{T_1+2+i}^*)$. By the above results, we have

$$\sum_{i=0}^{\infty} \beta^i \Pi(p_{T_1+2+i}^*, r_{T_1+2+i}^*) \leq \sum_{i=0}^{\infty} \beta^i \left( p_{T_1+2+i}^* D_1(p_{T_1+2+i}^*) + p_{T_1+2+i}^* D_2(p_{T_1+2+i}^*, r_{T_1+2+i}^*) \right)$$

$$\leq \sum_{i=0}^{\infty} \beta^i \left( \frac{a_1^2}{4b_1} + \frac{a_2^2}{4b_2} + \frac{a_1}{b_1} \gamma(r_{T_1+2+i}^* - p_{T_1+2+i}^*) \right)$$

$$\leq \frac{1}{1 - \beta} \left( \frac{a_1^2}{4b_1} + \frac{a_2^2}{4b_2} + \frac{2a_1 \gamma \epsilon}{b_1} \right)$$

$$\leq \frac{1}{1 - \beta} \left( \frac{a_1^2}{2b_1} + \frac{a_2^2}{4b_2} \right) = C_1$$

where the last inequality is by the fact that $\gamma \epsilon \leq a_1/8$.

We next show that $r_{T_1+2}^* \leq (1 - \theta)a_1/2b_1$. Otherwise, consider the following policy $p_t'$: from $t = 0$ to $T_1 + 1$, $p_t' = p_t^*$; for $t = T_1 + 2$, $p_t' = \min\{(1 - \theta)a_1/4b_1, a_2/2b_2\}$; for $t \geq T_1 + 3$, $p_t' = 0$. Clearly, the two policies $p_t^*$ and $p_t'$ generate the same revenue before period $T_1 + 2$. Now consider the revenue generated in $T_1 + 2$ for the policy $p_t'$. Because $p_{T_1+2}^* \leq r_{T_1+2}^* = r_{T_1+2}'$, we have

$$\Pi(p_{T_1+2}', r_{T_1+2}') \geq p_{T_1+2}' \left( a_2 - b_2 p_{T_1+2}' + \gamma \left( \frac{(1 - \theta)a_1}{2b_1} - p_{T_1+2}' \right) \right)^+.$$  \hfill (12)

If $(1 - \theta)a_1/4b_1 \leq a_2/2b_2$, then $p_{T_1+2}' = (1 - \theta)a_1/4b_1$ and

$$\text{RHS of (12)} \geq \frac{(1 - \theta)a_1}{4b_1} \left( \gamma \frac{(1 - \theta)a_1}{4b_1} \right) \geq \gamma \frac{(1 - \theta)^2 a_1^2}{16b_1^2} > C_1.$$

If $(1 - \theta)a_1/4b_1 > a_2/2b_2$, then $p_{T_1+2}' = a_2/2b_2$ and

$$\text{RHS of (12)} \geq \frac{a_2}{2b_2} \left( \gamma \frac{a_2}{2b_2} \right) \geq \gamma \frac{a_2^2}{4b_2^2} > C_1.$$
In either way, the revenue generated in a single period \(T_1 + 2\) by the new policy \(p'_t\) is higher than \(C_1\), which is the total discounted revenue generated by the optimal policy from \(T_1 + 2\) onward. This contradicts the fact that \(p^*_t\) is optimal. Thus, we have proved that \(r^*_{T_1+2} \leq (1 - \theta)a_1/2b_1\). Therefore, \(p^*_{T_1+2} \leq r^*_{T_1+2}/(1 - \theta) + 2\epsilon/(1 - \theta) \leq a_1/2b_1 + 2\epsilon/(1 - \theta) \leq 3a_1/4b_1\).

We next show that \(p^*_t\) cannot be optimal. Consider the following policy \(p'_t\): from \(t = 0\) to \(T_1 + 1\), \(p'_t = p^*_t\); for \(t = T_1 + 2\), \(p'_t = a_1/b_1\); for \(t = T_1 + 3\), \(p'_t = \min \{(1 - \theta)a_1/8b_1, a_2/2b_2\}\); for \(t \geq T_1 + 4\), \(p'_t = 0\). Now we compute the revenue generated in period \(T_1 + 3\). We have

\[
r'_{T_1+3} = (1 - \theta)p'_{T_1+2} + \min_{0 \leq t \leq T_1+2} \{p'_t\} \\
\geq (1 - \theta)p^*_{T_1+2} + (1 - \theta) \left( \frac{a_1}{b_1} - p^*_{T_1+2} \right) + \min_{0 \leq t \leq T_1+2} \{p^*_t\} \\
\geq r^*_{T_1+3} + \frac{(1 - \theta)a_1}{4b_1}.
\]

Then the revenue in period \(T_1 + 3\) is

\[
\Pi(p'_{T_1+3}, r'_{T_1+3}) \geq p'_{T_1+3} \left( a_2 - b_2p'_{T_1+3} + \gamma \left( \frac{(1 - \theta)a_1}{4b_1} - p'_{T_1+3} \right) \right).
\] (13)

If \(a_2/2b_2 \leq (1 - \theta)a_1/8b_1\), then \(p'_{T_1+3} = a_2/2b_2\) and

\[
\text{RHS of (13)} \geq \gamma \frac{a_2^2}{4b_2^2} > \beta^{-1}C_1.
\]

If \(a_2/2b_2 > (1 - \theta)a_1/8b_1\), then \(p'_{T_1+3} = (1 - \theta)a_1/8b_1\) and

\[
\text{RHS of (13)} \geq \gamma \frac{(1 - \theta)^2a_1^2}{64b_2^2} > \beta^{-1}C_1.
\]

Either way, the revenue generated by \(p'_t\) in period \(T_1 + 3\), discounted to \(T_1 + 2\), is higher than \(C_1\), the total discounted revenue generated by the optimal policy from \(T_1 + 2\) onward. Because \(p'_t\) and \(p^*_t\) generate the same revenue before \(T_1 + 2\), it contradicts the fact that \(p^*_t\) is the optimal policy. Therefore, we have proved that \(p^*_t\) does not admit a steady state. ■