Selling Dreams: Pricing under Anticipation

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Frequent sales are a characteristic of the modern retail industry despite their significant implementation costs and potential to erode consumers’ price expectations. One reason for this is that consumers enjoy looking forward to sales. We model the purchase behavior of “emotionally rational” consumers, and explain their appeal for sales, by accounting for anticipatory feelings triggered by the prospect of buying at a discount, as well as for the disappointment when anticipated outcomes fail to materialize. At the customer level, our model explains why people overspend—that is, purchase at prices above their valuation—when firms offer sales. As a consequence, sales policies can outperform uniform pricing when a monopolist sells to anticipating customers. In fact, both firm and customers benefit from these anticipatory feelings when firms vary prices. Our model also explains why, in a competitive setting, a firm cannot benefit from adopting uniform pricing when its competitors offer sales. Our results help explain the recent failure of US-based retailers JC Penney and Macy’s to sustain an everyday low price (EDLP) policy.

Key words: anticipation, loss aversion, disappointment, sales policy, every day low price policy, retail industry

1. Introduction

We don’t simply think about future possibilities; we feel future possibilities.

—David Huron

A pair of your favorite brand of running shoes is on display in a department store but the price is high for your budget. You later get a coupon in the mail advertising 30% off selected items in that store. Although unsure about whether it can be used on the shoes, you begin to imagine yourself fortunate to get the shoes on sale, running in their comfort, and clocking faster laps. The rosy picture conjures up positive emotions even though no material transaction has been concluded. Savoring these emotions, if sufficiently strong, makes you feel attached and somewhat entitled to the shoes. Yet this dream may not materialize if you find, when visiting the store, that the shoes are at their regular, high price. At this point, however, leaving the store empty-handed would only increase your disappointment, which you may avoid by buying the shoes despite their high
price. Interestingly, this purchase would likely not have been made if you had not been lured into dreaming about the shoes by the prospect of a sale.

Anticipating a positive (or negative) outcome causes feelings of joy (or dread) before a prospect is actually realized. The anticipated outcome, such as purchasing at a discount, acts as a benchmark against which actual purchase outcomes are compared. This ex post comparison may lead to disappointment (or elation) if the realized outcome compares unfavorably (or favorably) to the anticipated one. For example, a consumer may feel disappointed if she anticipates buying on sale and ends up not getting a deal; such disappointment would not occur, however, if she did not anticipate buying. We seek to understand how anticipatory feelings affect consumers’ purchase decision-making process and the firm’s pricing policies.

We develop a general framework for sequential purchase decisions under anticipation. Our model of consumer behavior accounts for anticipatory feelings—that is, the ex ante emotions triggered by savoring an uncertain prospect such as a discount purchase. Consumers’ experiences due to anticipation are known as “conceptual consumption” (Ariely and Norton 2009); an important feature of our model is that it incorporates the utility from conceptual consumption into consumers’ rational decision-making process.

More specifically, the consumer who faces an uncertain prospect may anticipate different outcomes, each resulting in a different expected surplus. In line with evidence that anticipation activates reward centers in the brain (Ernst et al. 2004), part of consumer surplus is assumed to derive from sheer anticipation. In our setup, the consumer is emotionally rational (aka emorational, Oullier 2010) in that he chooses an anticipatory outcome that maximizes his total (material plus emotional) expected surplus. In contrast, an optimistic consumer hopes for the best outcome while a pessimist dreads the worst. Although anticipating the most favorable outcome (e.g., a discount purchase) provides the consumer with maximum positive anticipatory feelings, it also increases the chances of experiencing disappointment once the uncertainty is resolved. An emotionally rational customer must therefore balance the felicity from anticipating a bargain purchase with the potential disappointment ensuing should the dream deal fall through.

In this context, then, our first research question is: How does anticipation as an ex ante emotion interact with the ex post emotions of elation and disappointment to shape the consumers’ purchase decision-making process? For example, we seek to explain behavioral regularities such as overspending, or why some consumers spend more than they have budgeted when anticipating the possibility of buying on sale, and self-delusion, as when consciously preparing not to buy, regardless of price, as a mechanism to limit disappointment and splurging.

We seek also to understand how firms can use pricing policies that account for consumer behavior under anticipation. In particular, we investigate when and to what extent uncertainty in prices
benefits the firm and anticipative consumers. One the one hand, research suggests that firms should offer constant, steady-state prices to loss-averse consumers because the negative effect of perceived surcharges outweighs the benefit of discounts (Heidhues and Koszegi 2005, Popescu and Wu 2007). On the other hand, the (uncertain) prospect of sales allows consumers to dream about getting a deal—which may explain the prevalence of sales in practice (Ellickson et al. 2012). These considerations motivate our second research question: When is it profitable for a firm facing anticipative consumers to offer sales rather than constant pricing policies?

Even if no less valuable than sales and promotions in terms of material surplus, offering constant prices could be a potential setback for the firm if doing so takes away the pleasure customers derive from hunting discounts. Indeed, we find that a monopolist is better-off with a random sales policy whenever loss-averse consumers’ anticipatory feelings exceed the disappointment from unmet expectations. In this case, the full price charged by the firm exceeds consumer’s valuation of the product, and the optimal frequency of discounts induces the consumer to visit the store and buy even if the price is high (i.e., above her valuation). The reason is that anticipating a discount purchase triggers feelings of attachment to the product, which leads to disappointment if the purchase is forgone; hence the consumer would rather pay more than his valuation in order to avoid such disappointment. We show that this sales policy strictly outperforms a uniform-price policy. Interestingly, customers are also better-off: they derive higher overall expected surplus under a random rather than a constant price policy. This is because the sales policy allows them to enjoy anticipating a purchase at a discount, which leaves strictly positive surplus; in contrast, the optimal single-price policy leaves consumers no surplus.

Finally, given the recent attempts of retailers (e.g., JC Penney, Macy’s) to institute an everyday low price (EDLP) regime, we use our model to explain why unilaterally offering a fixed-price policy may be ineffective when competitors are offering sales. Our results suggest that this is the case when consumers’ anticipatory feelings are sufficiently strong. Starting from the premise that saving (i.e., buying cheaper) is why customers respond to sales, JC Penney fixed prices at 40% lower than initial list prices before EDLP; thus customers were effectively guaranteed the same savings (Wall Street Journal 2012). What this strategy failed to recognize is that “most shoppers . . . want the thrill of getting a great deal, even if it is an illusion”; as a result, “customers rebelled, traffic declined, sales fell and Penney slowly returned to . . . pricing with lots of promotions, lots of price-focused ads, and marked-up prices that would be marked-down” (New York Times 2013).1

The rest of the paper is organized as follows. In Section 2, we review the related literature. Section 3 formalizes the consumer behavior model under anticipation. In Section 4, we study the behavioral

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1 Similar consumer behavior also resulted in Macy’s abandoning its attempts to reduce coupon offers (New York Times 2007).
regularities that can be explained by anticipation and characterize the consumers’ optimal response to a given price policy. Section 5 studies a monopolist’s optimal price policy and explores EDLP policy performance in a competitive market with anticipating consumers. Section 6 concludes.

2. Literature Review

The modeling literature on anticipation is relatively sparse. The earliest economic model of anticipation is proposed by Loewenstein (1987), who studies how an individual values a single certain future outcome over time. He shows that the individual may hasten a negative experience (a dental appointment) to shorten the period of dread and may postpone a positive experience (taking a vacation) to lengthen the period of savoring. Baucells and Bellezza (2015) propose the “Anticipation and Recall” model that integrates the instant utilities experienced before, during, and after an event to construct a total utility profile for that event. This profile, during anticipation, is U-shaped, suggesting that there is an optimal duration of anticipation. We abstract from the temporal effects of anticipation so as to focus on consumer anticipative behavior under (price) uncertainty.

Caplin and Leahy (2001) show that anticipation may result in time inconsistency which suffices to explain the equity premium puzzle. Gollier and Muermann (2010) and Jouini et al. (2014) develop a theory of decision making under risk (e.g., portfolio choice) that accounts for both anticipatory feelings and disappointment. In these models, the customer chooses a degree of optimism by distorting the objective probabilities that balance the pleasure from anticipation ex ante and negative feelings ex post; the anticipated payoff is then the subjective certainty equivalent of the risky prospect. These models successfully explain such behavioral regularities as the Allais Paradox. One limitation of these models is that they do not allow for making a decision after uncertainty is resolved. This feature is key in our model: once the price is realized, the customer must decide whether or not to buy. Moreover, our customer anticipates an actual feasible outcome (e.g., full or discount price) and not a (subjective) certainty equivalent thereof. This approach is consistent with that of He et al. (2013), although their paper does not specify a mechanism for selecting the anticipated outcome.

Kőszegi (2010) develops a model of anticipation in which beliefs and behavior are jointly determined in a “personal equilibrium”; that is, the behavior given past beliefs must be consistent with those beliefs. In Kőszegi’s model, unlike ours, the decision maker does not have direct control over his expectations. In other words, he only makes choices over physical outcomes that in turn affect his expectations. In contrast—but in accordance with the “optimal belief” approach adopted by Brunnermeier and Parker (2005), Gollier and Muermann (2010), and Jouini et al. (2014)—our decision maker chooses what optimally to anticipate and thereby balances ex ante savoring and ex post disappointment. This modeling choice allows us to explain behavioral regularities such as overspending and inconsistencies between action and anticipation.
Our paper also contributes to a vast literature on sales and promotions that examines why such policies are effective with customers. The attractiveness of sales is attributed to hedonic or utilitarian benefits to customers (for a review of the literature and concepts, see Chandon et al. 2000), and monetary promotions are typically more effective for utilitarian products (such as those offered by JC Penney and Macy’s) than for hedonic products (e.g., so-called experience goods). Liu (2014) suggests that sales are attractive because of “utility blindness”: customers fail to process all the information related to a purchase; thus they end up considering only the “transaction utility” (i.e., gains from the deal) instead of the total utility, which also includes “acquisition utility”.

Our results provide a new perspective that sales are optimal because they trigger anticipatory feelings in consumers; that is, consumers savor the anticipation of getting a deal, which prompts them to visit the store and (perhaps) to buy at higher prices. We show that both firm and consumers are better-off under a sales policy than under an EDLP policy. This paper is not unique in characterizing an optimal randomized price policy in a monopoly setting. Heidhues and Kőszegi (2014) demonstrate that a monopolist facing expectations-based loss-averse customers may find it optimal to randomize prices. However, they also show that implementing such a pricing policy requires market power and that, in a competitive market, deterministic prices perform better. This is because their customers dislike uncertain prices that are perceived as being manipulative and so would rather buy from a competitor offering a fixed price. In contrast, our consumers actually derive pleasure from anticipation; the implication is that even in a competitive market, a firm unilaterally offering EDLP can be outperformed by a competitor that adopts a sales policy. Our model thus provides theoretical support for the empirical findings that (i) promotional pricing strategies yield higher revenues than EDLP and (ii) US retailers that adopted promotional pricing strategies in response to the entry of Walmart fared better than those adopting an EDLP policy (Ellickson et al. 2012).

Sales policies have been studied extensively in the operations management literature. In a market with strategic consumers, Cachon and Feldman (2015) show that it may be optimal for a firm to commit to frequent discounts in order to attract more of the high-valuation customers, who then choose to incur the search cost of looking for sales; that paper includes an overview of the related literature. Besbes and Lobel (2015) explain cyclic pricing by assuming that strategic consumers have varying degrees of patience. Our paper differs in that we focus on the behavioral motives for offering random sales; also, our results are driven neither by valuation heterogeneity nor by consumer patience. So even though our consumers are forward looking in anticipating their emotions, they do not wait for discounts.

Our paper is the first in the behavioral operations literature to offer a rigorous model of sequential decision making under anticipation and then to explore its operational consequences. The
behavioral pricing literature focuses mainly on the emotions experienced after material payoffs have been realized and how those emotions compare with a context-specific reference point. For example, Popescu and Wu (2007) and Nasiry and Popescu (2011) assume that consumers form history-dependent reference prices, which—in the spirit of prospect theory (Kahneman and Tversky 1979)—cause feelings of gain or loss. In these papers, consumers are unaware of their bias and do not act strategically. The major finding is the optimality of constant pricing in the long run when buyers are loss averse. Liu and Shum (2013) and Baron et al. (2015) assume that consumers are disappointed when their expectations of price and product availability are not met; these authors study the firm’s pricing and inventory decisions and characterize the conditions under which consumers’ aversion to disappointment may benefit the firm.

Much as in our work, Nasiry and Popescu (2012) and Özer and Zheng (2015) assume consumers to be emotionally rational and therefore anticipating their subsequent emotions when optimizing their consumption plans. Nasiry and Popescu find that firms should not advance sell if consumers anticipate sufficient regret from buying versus not buying a product when their valuation is uncertain. Özer and Zheng show that consumers’ regret and misperception of availability may render markdowns preferable to EDLP. Our paper differs from these papers in that consumers here do not regret their decisions—though they do feel disappointed when their expectations do not materialize. More importantly, our work is distinctive in allowing consumers not only to think about future possibilities but also to “feel” them ex ante; for that purpose we incorporate the utility derived from anticipation into our analysis.

3. A Model of Consumer Behavior with Anticipation

In this section, we model the behavior of a forward-looking customer subject to anticipation in a market with price uncertainty. Given prices and the frequency of sales, we characterize the consumer’s decision process as well as her total emotional and material surplus. We then use this model to derive consumer purchase decisions under anticipation (in Section 4) and the firm’s optimal pricing policy (in Section 5).

Assume that the market consists of a representative consumer with a known valuation $v$. The firm implements a sales policy $\tilde{p} = (p_l, p_h; q)$ that consists of offering the product at a discount price $p_l$ with probability $q$ and at full price $p_h$ with probability $1 - q$. In particular, a fixed-price policy corresponds to $q = 1$ or $p_h = p_l$. The customer is aware of the price distribution $\tilde{p}$, but does not know the actual price prior to visiting the store.

Recalling the shoe purchase example from the Introduction, before the store visit you might picture yourself snapping up the shoes at a discount $p_l$ or might instead choose to keep your expectations under control by anticipating a high price $p_h$. Once you are in the store, you see
the actual price and must decide whether to buy or to leave empty-handed. However, if you had already pictured yourself in the high-flying shoes, then leaving the store without them would be a disappointment. We next formalize this customer decision-making process, which evolves in two stages.

In the first stage—before visiting the store—the consumer anticipates what price will materialize and whether (or not) he will purchase the product at that price. Once the price is realized, the customer decides whether or not to actually make a purchase. The anticipated outcome will act as a reference point in this decision, relative to which actual outcomes are compared. So when selecting which outcome to anticipate, the “emo-rational” customer must balance the felicity from anticipating this outcome with the potential ensuing disappointment should that outcome not transpire. In sum, the consumer derives utility from three dimensions: anticipation (conceptual consumption) in period 1, material utility from a purchase in period 2, and the reference-dependent utility associated with the discrepancy between what is anticipated in period 1 and the reality of period 2. We shall describe each of these components and their corresponding expected surplus in our context. Figure 1 illustrates the consumer decision-making process.

**Utility from Anticipation.** Four anticipation levels are possible, one for each state of the world (i.e., for each price–purchase decision pair). Specifically, the consumer can anticipate either (a) that he gets the discount and then buys, \((p_l,1)\), or does not buy, \((p_l,0)\); or (b) that he does not get the discount and then buys, \((p_h,1)\), or does not buy, \((p_h,0)\). Let \(A = \{(p_a,x_a) \mid p_a \in \{p_l,p_h\}, x_a \in \{0,1\}\}\) denote the outcome space, where \(x_a = 1\) for buying and \(x_a = 0\) for not buying. As the customer anticipates one of these four outcomes, he experiences anticipatory feelings before the uncertainty about the firm’s price is realized. For simplicity we assume the utility from anticipation
or $U^a(p_a, x_a)$ is proportional to the expected economic surplus of the anticipated outcome; that is, $U^a(p_a, x_a) = U^a(p_a, x_a; \hat{\tilde{p}}) = k x_a (v - p_a) \mathbb{P}(\hat{\tilde{p}} = p_a)$, where $k$ captures the strength of anticipatory feelings. Consistent with Brunnermeier and Parker (2005), Gollier and Muermann (2010), and Jouini et al. (2014) the more likely the anticipated outcome, the stronger the value derived from anticipation. Note that a consumer who does not expect to consume ($x_a = 0$) derives no utility from anticipation ($U^a = 0$).

**Economic Utility.** In the second stage—when the price $p$ has been revealed—the consumer decides whether or not to buy at that price. If she does then her material surplus is $v - p$, but otherwise it is zero; overall, then, her economic surplus is $U(p, x) = (v - p) x$.

**Gain–Loss Utility.** When reality does not match anticipated outcomes, consumers experience disappointment. This can happen along two dimensions: consumption and money. If the customer anticipates a purchase but decides ex post to forgo it (e.g. because the actual price is too high), then he experiences a feeling of loss from being deprived of the anticipated consumption value $v$ (Kőszegi and Rabin 2006, 2009, Courty and Nasiry 2015). Furthermore, a customer who anticipates a purchase at the discount price but eventually buys at full price experiences a perceived monetary loss $p_l - p_h$. We assume that customers are loss averse; in other words, the pain due to perceived losses is greater than the pleasure due to gains of equal magnitude. In our setting, this means that the disappointment from paying more than anticipated exceeds the elation from an unexpected deal. For simplicity, we use a reference-dependent value function that is piecewise linear and also flat in the gain domain for both consumption and money. Given an anticipation level ($p_a, x_a$), the disappointment component of customer’s utility corresponding to a price $p$ and purchase decision $x \in \{0, 1\}$ is given by:

$$U^r(p, x; p_a, x_a) = u^r(p - p_a) x + u^r(x - x_a) v \quad \text{where} \quad u^r(y) = \begin{cases} 0 & \text{if } y \geq 0 \\ \lambda y & \text{if } y < 0 \end{cases},$$

and $\lambda > 0$ is the loss-aversion coefficient. The consumer experiences a gain/loss in money only after making a purchase (i.e., only if $x = 1$). To simplify the presentation, we focus on disappointment and ignore its positive counterpart, elation. However, we can extend the model to account for elation by introducing a positive slope for gains that is less steep than $\lambda$ (for loss aversion).

In sum, there are three components to customer’s total surplus: the standard economic surplus $v - p$ from making a purchase, the reference-dependent component $U^r$ corresponding to gain/loss perceptions relative to the anticipated outcome, and the ex ante utility $U^a$ from savoring (or dreading) the anticipated outcome. The last component is unique to our model and enters only the first stage decision. Formally: given an anticipated outcome ($p_a, x_a$), the maximum expected surplus that the consumer derives from shopping is

$$W^a(p_a, x_a) = U^a(p_a, x_a) + \mathbb{E}_{\hat{\tilde{p}}} \left[ \max_{x \in \{0, 1\}} \{(v - p) x + U^r(p, x; p_a, x_a)\} \right].$$

(1)
The consumer in our model is rational about his emotions (or emo-rational) and forward looking in his stochastic purchasing decision problem. He therefore selects his anticipation level \((p_a, x_a) \in A\) optimally so as to maximize the total expected surplus—both material and emotional (savoring and disappointment)—from his shopping experience. Once the price \(p \in \{p_l, p_h\}\) is realized (according to distribution \(\tilde{p}\)), the customer decides whether or not to buy at this price. That decision (and corresponding surplus) are reflected in the initial choice of anticipation level which serves as a reference point relative to which actual outcomes are compared. So in selecting which outcome \((p_a, x_a)\) to anticipate, the consumer must balance the felicity from savoring the anticipation of this outcome with the disappointment ensuing if that outcome does not materialize. The customer’s decision problem can thus be modeled as a two-stage stochastic program with recourse:

\[
W = \max_{(p_a, x_a)\in A} w_a(p_a, x_a) = U^a(p_a, x_a) + \mathbb{E}_{\tilde{p}} \left[ \max_{x \in \{0, 1\}} \{ (v - p) x + U^r(p, x; p_a, x_a) \} \right].
\] (2)

In particular, if the firm charges uniform prices (i.e., \(p_l = p_h = p\) and \(q = 1\)) then the only possible price to anticipate is \(p_a = p\), in which case \(W(p, 1) = kq(v - p)\) and \(U^a(p_l, 1) = kq(v - p)\) and she enjoys that utility before the uncertainty is resolved. Once the price becomes known, the consumer must actually decide whether or not to buy. If the discount is offered then buying yields the economic surplus \(v - p_l\); not buying results in perceived loss due to nonconsumption, \(-\lambda v\). Yet if the firm offers the product at the full price, then a decision to purchase induces not just the consumption utility \(v - p_h\) but also feelings of loss in the money dimension, \(-\lambda(p_h - p_l)\), because the customer had anticipated a discount. Not buying, however, results again in a perceived loss of \(-\lambda v\) relative to anticipated consumption. Putting everything together, we obtain the customer’s expected surplus from anticipating to buy at a discount, \((p_l, 1)\), as

\[
W(p_l, 1) = kq(v - p_l) + q \max\{v - p_l, -\lambda v\} + (1 - q) \max\{v - p_h - \lambda(p_h - p_l), -\lambda v\}. \tag{3}
\]

**Case 1.** If the customer anticipates a purchase at the discount price, \((p_a, x_a) = (p_l, 1)\), then her utility from anticipating this outcome is \(U^a(p_l, 1) = kq(v - p_l)\) and she enjoys that utility before the uncertainty is resolved. Once the price becomes known, the consumer must actually decide whether or not to buy. If the discount is offered then buying yields the economic surplus \(v - p_l\); not buying results in perceived loss due to nonconsumption, \(-\lambda v\). Yet if the firm offers the product at the full price, then a decision to purchase induces not just the consumption utility \(v - p_h\) but also feelings of loss in the money dimension, \(-\lambda(p_h - p_l)\), because the customer had anticipated a discount. Not buying, however, results again in a perceived loss of \(-\lambda v\) relative to anticipated consumption. Putting everything together, we obtain the customer’s expected surplus from anticipating to buy at a discount, \((p_l, 1)\), as

**Case 2.** Consider now the case \((p_a, x_a) = (p_l, 0)\), where the customer anticipates that a discount \(p_l\) will materialize and that he will nevertheless not buy at this price. If he does get the discount but decides instead to purchase, then he enjoys the economic surplus \(v - p_l\) (whereas not buying yields zero surplus). However, if the product sells at full price \(p_h\) and the customer decides to
purchase, then he derives the economic surplus $v - p_h$ but suffers a relative loss of $-\lambda(p_h - p_l)$ because the price actually paid is more than anticipated. We can therefore write the consumer’s expected surplus from anticipating not to buy at a discount

$$W(p_l, 0) = q \max \{v - p_l, 0\} + (1 - q) \max \{v - p_h - \lambda(p_h - p_l), 0\}. \quad (4)$$

**Case 3.** Similarly, when the consumer anticipates that the full price will materialize and that he will not buy, $(p_a, x_a) = (p_h, 0)$, then he derives no utility from anticipation yet suffers no disappointment:

$$W(p_h, 0) = q \max \{v - p_l, 0\} + (1 - q) \max \{v - p_h, 0\}. \quad (5)$$

**Case 4.** Finally, if the consumer anticipates buying at full price, $(p_a, x_a) = (p_h, 1)$, then her utility from anticipating this outcome is $U_a(p_h, 1) = k(1 - q)(v - p_h)$. Observe that this is negative when $p_h > v$, in which case the consumer dreads this prospect. Regardless of the selling price $p$, her economic surplus from buying is $v - p$ but if not buying she suffers a perceived consumption loss $-\lambda v$. Formally,

$$W(p_h, 1) = k(1 - q)(v - p_h) + q \max \{v - p_l, -\lambda v\} + (1 - q) \max \{v - p_h, -\lambda v\}. \quad (6)$$

### 4. Regularities in Purchase Behavior under Anticipation

In what follows we aim to characterize the customer’s choice of anticipated outcome, as determined by model (2), and how this anticipation affects his subsequent purchase decisions. In particular, we aim to evidence behavioral regularities, such as: (i) when a customer is inconsistent, in that his actual purchase decision is other than he anticipated; and (ii) when the customer makes purchases that yield a negative economic surplus.

For consumers to buy at a negative surplus, it is necessary that one (but not both) of the prices exceeds valuation: $p_l \leq v \leq p_h$. Indeed, if $v \leq p_l \leq p_h$, then consumers will not purchase regardless of the price because (a) doing so would result in a negative surplus and (b) they will correctly anticipate this decision. If $p_l \leq p_h \leq v$, then a positive surplus results from buying at any price; hence consumers will always anticipate buying, and do so. So neither of these cases leads to behavioral regularities. We therefore focus our analysis on the relevant scenario by making, without loss of generality, the following assumption.

**Assumption 1.** Prices are such that $p_l \leq v \leq p_h$.

Under Assumption 1, the customer’s surplus corresponding to each anticipation level simplifies as follows:

$$W(p_l, 0) = W(p_h, 0) = q(v - p_l), \quad (7)$$

$$W(p_h, 1) = k(1 - q)(v - p_h) + q(v - p_l) + (1 - q) \max \{v - p_h, -\lambda v\}, \quad (8)$$

$$W(p_l, 1) = q(k + 1)(v - p_l) + (1 - q) \max \{v - p_h - \lambda(p_h - p_l), -\lambda v\}. \quad (9)$$
In particular, when she does not anticipate buying \((x_a = 0)\), the customer derives no utility from anticipation (and experiences no disappointment because she only buys at a discount). Her expected surplus is the same regardless of the anticipated price and exceeds that from anticipating a purchase at full price, \(W(p_h,0) \geq W(p_h,1)\), which thus cannot be a viable anticipated outcome.

**Lemma 1.** Under Assumption 1, anticipating to buy at full price \((p_h, 1)\) cannot be optimal.

Because \(p_l \leq v\), the consumer always buys at low price irrespective of what she anticipates. However, she would never buy at full price \((p_h \geq v)\) unless (i) she anticipated doing so at a discount, \((p_a, x_a) = (p_l, 1)\), and (ii) the monetary pain from paying full price is less than the pain from forgone consumption: \(v - p_h - \lambda(p_h - p_l) > -\lambda v\), or, equivalently, \(v > p_h - \frac{v}{1+\lambda}p_l\) (see equation (9)). In that case, the customer is prone to a pseudo-endowment effect (Prelec 1990) whereby the customer pays full price (more than her valuation) simply to avoid the consumption disappointment of forgoing the anticipated purchase. We next investigate the circumstances (if any) under which such apparently irrational behavior is likely to occur and may even be optimal.

The following result characterizes the customer’s optimal anticipation level and decision in response to a given pricing policy \((p_l, p_h, q)\).

**Proposition 1.** Suppose a pricing policy \((p_l, p_h, q)\) satisfies Assumption 1. If \(k \leq \frac{1-q}{q} \min \left\{ \frac{v}{v-p_l} \lambda, (1+\lambda)\frac{p_h-v}{v-p_l} + \lambda \right\}\), then it is optimal for the consumer to anticipate not buying (i.e., \(x_a = 0\)) and he only purchases at a discount. Otherwise, the optimal anticipation level is buying at a discount (i.e., \((p_a, x_a) = (p_l, 1)\)). In this latter case, if \(v > p_h - p_l\) and \(\lambda \geq \frac{p_h-v}{p_l-p_h+v}\) then the consumer buys in both states; otherwise, he only buys at a discount.

Figure 2 plots consumer purchase behavior under anticipation according to Proposition 1. Introducing anticipation into the customer’s decision process raises several questions. First, is it possible that consumers anticipate buying at a certain price and then, when that price materializes, they actually decline to buy? Conversely, do consumers anticipate not buying at a given price and then actually do buy at that price? Second, is it possible that anticipation makes consumers overspend—that is, buy at prices above their valuations? Proposition 1 identifies these two types of behavioral regularities, and characterizes when they occur.

First, a consumer may appear inconsistent, in that she need not carry through her anticipated purchase decisions, even if the anticipated price materializes. Specifically, when disappointment dominates anticipatory feelings (i.e., when \(\lambda\) is large relative to \(k\); see the subdiagonal regions in Figures 2(a) and 2(b)), we find that consumers consciously refrain from anticipating to buy on sale even as they eventually end up doing so. In this case, anticipating not to purchase acts as a
form of self-control: it precludes disappointment when the sale does not materialize. Such nonpurchase anticipation also avoids the ensuing cost of being trapped into overspending by consumption expectations, as explained next.

Indeed, overspending is the second behavioral regularity explained by our model. A sufficiently emotional consumer may anticipate purchasing at a discount, \((p_l, 1)\), and then end up buying even if prices exceed his valuation. Proposition 1 shows this to occur when anticipatory feelings dominate disappointment and both emotions are relatively strong (the shaded area in Figure 2(a)). Such a consumer derives so much pleasure from the prospect of a discount purchase that he will always choose to anticipate it—at the cost of being disappointed if the deal does not materialize. Moreover, because the disappointment from not buying (and hence not consuming) would be so great in that event, he would rather avoid it by paying the full price. Such overspending makes (emotional) sense provided the full price is not too high, \(p_h \leq v + p_l\). Otherwise, the cost of avoiding disappointment is just too large and so the consumer’s rationality prevails over his emotions (Figure 2(b)).

It is interesting that this result persists even in absence of monetary loss aversion. It is the combination of anticipation and consumption disappointment that drives such behavior. If the customer were to feel no disappointment when forgoing an anticipated purchase (i.e., if there were no loss aversion with respect to consumption), then he would not buy at full price; mathematically, equation (3) would become

\[
W(p_l, 1) = kq(v - p_l) + q \max \{v - p_l, 0\} + (1 - q) \max \{v - p_h - \lambda(p_h - p_l), 0\} = kq(v - p_l) + q(v - p_l).
\]
Similarly, in the absence of anticipation a customer is better-off not anchoring on consumption and purchasing only if the low price is realized. Indeed, if \( k = 0 \) then equations (3) and (4) would imply, under Assumption 1, that \( W(p_l, 1) < W(p_l, 0) = q(v - p_l) \). Thus buying above valuation is ruled out: anchoring on consumption would not yield any positive anticipatory utility, but would create feelings of loss upon nonconsumption.

The next result identifies necessary and sufficient conditions for consumers to buy at a negative economic surplus. By Proposition 1, a consumer who considers this possibility must first optimally anticipate purchasing at a discount; that is, it must be that \( W(p_l, 1) \geq W(p_h, 0) = W(p_l, 0) \), as we have already shown that anticipating to buy at high price is suboptimal, \( W(p_l, 1) \geq W(p_h, 1) \) (Lemma 1). Given the firm’s pricing policy, our next result characterizes when the consumer buys at a negative economic surplus and so formalizes the preceding discussion.

**Corollary 1.** Under Assumption 1, if \( p_h - v \leq \min \{ \frac{k + \lambda}{1 + \lambda} p_l; \frac{(k + \lambda)q - \lambda}{1 - q (1 + \lambda)} (v - p_l) \} \) then the customer anticipates buying on sale \((p_a, x_a) = (p_l, 1)\) and eventually buys at either realized price.

5. **Optimal Pricing Strategy**

The previous section characterized consumer purchase decisions under anticipatory feelings and identified conditions that lead customers to overspend. In this section, we identify conditions on behavioral parameters that allow firms to extract more than the consumer’s valuation and thus to increase their profits by leveraging her anticipative emotions. We shall also characterize the firm’s optimal policy and profit before providing sensitivity results with respect to behavioral characteristics.

5.1. **Conditions for Optimality**

Given the consumer behavior characterized in Proposition 1 and a marginal cost of \( c > 0 \), the firm sets prices \( p_l, p_h \geq 0 \) and the sales frequency \( q \in [0, 1] \) to maximize its profits \( \pi(p_l, p_h, q) = qp_l x_l^* + (1 - q) p_h x_h^* - c \). Here \((x_l^*, x_h^*) \in \{0, 1\} \times \{0, 1\}\) are the optimal customer purchase decisions given the firm’s pricing policy and the customer’s optimal anticipation level \((p_a, x_a)\):

\[
x_l^* = \arg\max_{x_l \in \{0, 1\}} \{(v - p_l)x_l + U^r(p_l, x_l; p_a, x_a)\},
\]
\[
x_h^* = \arg\max_{x_h \in \{0, 1\}} \{(v - p_h)x_h + U^r(p_h, x_h; p_a, x_a)\}.
\]

We first provide some necessary conditions for a sales policy \((p_l, p_h, q)\) to be optimal; these will allow us to reformulate the firm’s optimization problem. In particular, Assumption 1 is necessary for optimality.

**Lemma 2.** An optimal pricing policy \((p_l, p_h, q)\) must satisfy \( p_l \leq v \leq p_h \) and \( qp_l + (1 - q)p_h > v \).
To see this, note first that setting both prices below valuation, \( p_l \leq p_h < v \), is suboptimal because it results in less profit than does a uniform-price policy, \( p_l = p_h = v \), which yields \( \pi(v, v, 1) = v - c \). Second, charging both prices above valuation, \( v < p_l \leq p_h \), is also suboptimal; this follows from the surplus equations (3)–(6). With such high prices, anticipation results only in dread and so customers avoid anticipating a purchase (\( W(p_l, 1) \leq W(p_l, 0) \) and \( W(p_h, 1) \leq W(p_h, 0) \)). Furthermore, since \( v - p_h \leq v - p_l < 0 \) it follows that the customer does not purchase when those prices are realized. As a result, the firm’s profit is zero. We conclude that an optimal pricing policy must satisfy \( p_l \leq v \leq p_h \), or that Assumption 1 must hold. The optimal policy must also satisfy \( qp_l + (1 - q)p_h > v \), for otherwise it yields less profit than does a uniform pricing policy \( (p_h = p_l = v) \). Because it is sufficient to focus on such policies when determining the firm’s optimal prices, we can make the following assumption without loss of generality.

**Assumption 2.** Prices are such that \( v < qp_l + (1 - q)p_h \).

We next investigate what type of consumers the firm can lure into purchasing at prices above valuation \( p_h > v \). We provide necessary and sufficient conditions on behavioral parameters for the existence of a pricing policy that induces such behavior.

**Proposition 2.** (i) Under Assumptions 1 and 2, if customers anticipate buying at a discount \( (p_l, 1) \) and always purchase the product, then the pleasure from anticipation must dominate disappointment (i.e., \( k > \lambda + 1 \)). (ii) If \( k > \lambda + 1 \) then there always exists a pricing policy \( (p_l, p_h; q) \) satisfying Assumption 1 that induces consumers to anticipate \( (p_l, 1) \), and always consume.

In short, the firm can entice consumers into buying at a negative surplus whenever savouring anticipation exceeds disappointment. In particular, Proposition 2 shows that anticipation is a critical driver of overpurchasing behavior; indeed, such behavior would not manifest in the absence of anticipatory feelings (i.e., when \( k = 0 \)).

### 5.2. Optimal Monopolistic Policy

So far we have shown that a firm can entice customers with strong anticipatory feelings to buy at prices above valuation. We next calculate the maximum profit the firm can derive from exploiting this type of consumer behavior and identify the pricing policies capable of achieving that maximum profit. In particular, we are interested in establishing precisely when a sales policy is better than a fixed-price policy.

Throughout this section, we assume that \( k > 1 + \lambda \); otherwise, consumers would not buy above their valuations (Proposition 2) and so the maximum firm revenue would be \( v \). Hence the firm’s optimization problem can be equivalently set up to focus on policies that induce consumers to anticipate a discount purchase yet then to buy at either price:

\[
\max_{(p_l, p_h, q)} \quad qp_l + (1 - q)p_h - c
\]
subject to

\[ p_h - v \leq \frac{\lambda}{1 + \lambda} p_l, \quad (10) \]
\[ p_h - v \leq \frac{(k + \lambda)q - \lambda}{(1 - q)(1 + \lambda)} (v - p_l). \quad (11) \]

The two constraints characterize the conditions under which the customer visits the store and buys no matter the price; see Corollary 1.

Our next proposition characterizes the firm’s optimal pricing policy.

**Proposition 3.** If \( k > 1 + \lambda \) then it is optimal for the firm to employ a randomized pricing policy: \( q^\ast = \sqrt{\frac{\lambda}{k-1}} \in [0, 1) \), \( p_l^\ast = (1 - \sqrt{\frac{k-1}{k} - \sqrt{\lambda}}) v \) and \( p_h^\ast = \frac{\lambda}{1 + \lambda} p_l^\ast + v \), which yields the profit \( \pi^\ast(\lambda, k, v, c) = \frac{\lambda}{k(1 + \lambda)} (\sqrt{k - 1} - \sqrt{\lambda})^2 v + v > v - c \). If \( k \leq 1 + \lambda \) then the firm’s optimal profit is \( \pi^\ast = v - c \), which can be achieved by a fixed-price policy \( p_h = p_l = v \).

Several observations are in order here. First, the optimal revenue exceeds that from the best fixed-price policy: \( \pi^\ast + c = \frac{\lambda}{k(1 + \lambda)} (\sqrt{k - 1} - \sqrt{\lambda})^2 v + v > v \). Therefore, the revenue per customer is higher than the customer valuation \( v \). It follows that the random price policy does strictly better than a uniform pricing policy, under which the revenue per customer is at most \( v \), the random policy does strictly better. The source of extra revenue is the surplus from anticipation, and the firm extracts part of this surplus.

It is interesting that also customers are better-off under a random than a uniform pricing policy. Consumers get zero surplus from a uniform policy that prices the product at their valuation. However, the consumers’ expected surplus under the optimal random policy in Proposition 3 is positive: \( W(p_l, 1) \geq 0 \). Under the optimal random policy, the surplus due to anticipation is \( U^a(p_l, 1) = kq^\ast(v - p_l^\ast) = (1 - q^\ast)\frac{\lambda}{k} v > 0 \); see equation (3). Because under the optimal policy, the consumer consumes in either state, his expected economic surplus together with gain/loss surplus, by equation (3), is \( q^\ast(v - p_l^\ast) + (1 - q^\ast)(v - p_h^\ast - \lambda(p_h^\ast - p_l^\ast)) = v - q^\ast p_l^\ast - (1 - q^\ast)p_h^\ast - (1 - q^\ast)\lambda(p_h^\ast - p_l^\ast) < 0 \). Therefore, the surplus due to anticipation is high enough to leave the consumer with a net positive expected surplus overall. Simply put, a firm that adopts EDLP forgoes profit and reduces the surplus of customers by precluding their anticipatory feelings.

Finally, observe that a firm adopting EDLP (and so pricing its product at the customer valuation \( v \)) cannot survive when the marginal cost of serving a customer exceeds her valuation—that is, if \( c > v \). In contrast, a sales pricing policy allows firms to sell products whose marginal cost exceeds \( v \); this is possible because part of the cost is transferred to the customer through anticipation.
5.3. Sensitivity to Behavioral Parameters

Our next result illustrates how the firm’s optimal policy and profits change with respect to behavioral parameters.

**Proposition 4.** Assume that $k \geq 1 + \lambda$. Then: (i) the optimal frequency of discounts increases with $\lambda$ and decreases with $k$; (ii) $p_h - p_l$ increases with $\lambda$; and (iii) the optimal profit increases in $k$ and it is unimodal in $\lambda$.

Part (i) of the proposition shows that the more consumers are averse to disappointment, the less likely the firm is to charge the high price. Because the optimal pricing policy induces consumers to buy at either price, there is no disappointment in the consumption dimension. Therefore, offering the discount more frequently reduces the chances of disappointment in the monetary dimension. However, the frequency of sales decreases with the strength of anticipation: consumers’ strong proclivities for savoring enable the firm to induce the anticipation of a discount purchase even as it keeps discounts rare—just to let the customers dream of a great deal. This dynamic could explain why customers participate in lotteries despite extremely slim chances of winning (no matter if perceived objectively or subjectively): lottery organizers are selling a dream to customers, who buy into that dream by paying for the ticket. A similar mechanism supports consumer demand for conditional upgrades (Cui et al. 2015).

It is therefore unsurprising that stronger anticipatory feelings also yield higher firm profits; as $k$ increases, the firm charges the high price more frequently. This price need not be monotone in $k$ but revenue from full price purchases increases; that is, $(1 - q)p_h$ increases with $k$. As a consequence, the firm offers discounts less frequently and sales revenue decreases; that is, $qp_l$ decreases with $k$.

Unlike previous results in the literature (Popescu and Wu 2007, Nasiry and Popescu 2011, Liu and Shum 2013), in our case the optimal profit is not monotonic in consumer loss aversion $\lambda$: rather, optimal profit first increases and then decreases with $\lambda$. The reason is that, for small values of $\lambda$, the gap between the two prices is also small. This follows from equation (3), whereby a small $\lambda$ implies that the disappointment in the consumption dimension $-\lambda v$ is small and hence, for $-\lambda v \leq v - p_h - \lambda(p_h - p_l) \leq 0$ to hold, prices must be close to each other and also close to $v$. An increase in $\lambda$ allows for larger price gaps, which benefits the firm. Yet as $\lambda$ increases, the disappointment in the monetary dimension $\lambda(p_h - p_l)$ increases and so the firm must offer the discount price more often to offset this effect. The result is that firm profits decline when $\lambda$ becomes sufficiently large.

In particular, this result suggests that firms should make every effort to increase consumer’s anticipatory feelings but not so much their disappointment. There is actually an ideal, intermediate level of disappointment that is high enough to induce overspending by consumers yet low enough to allow profitable variance in prices.
5.4. EDLP versus Randomized Pricing in a Competitive Market

We established in Section 3 that a monopolistic firm can obtain strictly higher profit with a sales policy than with a uniform-price policy. So absent competition, a firm is better-off varying prices to allow for consumer anticipation. However, it is not obvious how a randomizing firm fares in a market that includes an incumbent with an EDLP policy. Here we show that a firm using an optimal sales policy can outperform a competitor that adopts an optimal EDLP policy when the market consists of consumers who savor—to a sufficient degree—the anticipation of future outcomes.

In particular, we assume that a firm adopts a uniform-price policy and charges a price $c \leq p \leq v$, where the marginal cost $c$ is assumed to be the same for each firm. We then characterize the conditions under which a firm with a sales policy outperforms the one with the uniform-price policy.

**Proposition 5.** In a competitive market with anticipation, if $k > \max\{\lambda, (1 + \lambda)\frac{v}{c} - 1\}$ then a firm that adopts a randomized price policy strictly outperforms a competitor that offers a uniform pricing policy $c \leq p \leq v$.

Proposition 5 states that if consumers’ anticipatory feelings are sufficiently strong—or if consumers are not too averse to disappointment—then an EDLP policy cannot outperform a sales policy. Otherwise, there is no sales policy that outperforms the uniform price policy. This is because a carefully set uniform-price policy leaves to consumers an amount of surplus that cannot be surpassed or even equalled by any sales policy without sufficiently strong anticipatory feelings.

It is interesting that the two firms can coexist in the market. In other words, there are conditions under which customers are indifferent between the two firms yet the randomizing firm obtains strictly higher profit per customer than does the uniform pricing firm. These conditions are specified next.

**Corollary 2.** Assume that $k > \max\{\lambda, (1 + \lambda)\frac{v}{c} - 1\}$. If either (i) $\lambda < k < \lambda + 1$ or (ii) $k > \lambda + 1$ and $p \leq \frac{1-\lambda\sqrt{\lambda/(k-1)}}{k(k+1)}v$, then customers are indifferent between a firm with an EDLP policy and a firm with a sales policy.

By Proposition 5, the assumption on $k$ in Corollary 2 guarantees the existence of a sales policy that outperforms the uniform price policy. In particular, this assumption implies that $\frac{1-\lambda\sqrt{\lambda/(k-1)}}{k(k+1)}v < c \leq p$. Part (i) of the corollary states that, for relatively low values of $k$ (i.e., $\lambda < k < \lambda + 1$), customers are indifferent between the two firms. For high values of $k$ (i.e., $k > \lambda + 1$), customers are indifferent between the two firms provided the uniform price is not too high—that is, as long as $p \leq \frac{1-\lambda\sqrt{\lambda/(k-1)}}{k(k+1)}v$. Yet if $k > \lambda + 1$ and $\frac{1-\lambda\sqrt{\lambda/(k-1)}}{k(k+1)}v < p \leq v$, then not only the firm with an optimal sales policy outperforms the firm with the uniform-price policy but also customers strictly prefer the randomizing firm.
6. Conclusions
This paper proposes a rigorous model that accounts for consumers’ anticipatory experiences in a behavioral pricing context. We find that price uncertainty is key to triggering anticipatory consumer response and may induce consumer overspending when anticipatory feelings dominate consumer disappointment. A firm can leverage this behavior to extract higher profit by offering random sales. Both the firm and emotionally rational consumers are actually better-off with sales compared with a uniform pricing policy, which may explain why sales are ubiquitous in the modern retail industry. Although firms should make every effort to increase consumer anticipatory feelings (e.g., through advertising), there is an ideal level of disappointment that is high enough to trigger overspending but low enough to allow profitable price variation. We also show that a firm adopting a uniform-price policy in a market with anticipating customers may forfeit its customers to competitors who offer sales. This observation offers a possible explanation for why the experiments of JC Penney and Macy’s with uniform pricing were not successful.

To the best of our knowledge, this paper is the first in the operations literature to study anticipation and its consequences for consumer purchase decisions and firms’ pricing policies. Our model is stylized and opens several opportunities for extensions and further research. For example, one could move beyond our two-point distribution for prices and explore continuous price distributions (see Heidhues and Köszegi 2014). Doing so would require significant modifications in our consumer behavior model and analysis, although we expect our main insights (regarding the optimality of sales) are robust to that modification. Also, the misperception of (or subjective beliefs) about probabilities (Gollier and Muermann 2010, Özer and Zheng 2015) can influence anticipatory experiences and hence the effectiveness of sales policies. As long as those beliefs strengthen ex ante anticipation and consequently ex post disappointment, our results will carry through. Even so, it would be interesting to explore firms’ incentives to manipulate such beliefs. The goal of developing a fuller picture of consumer anticipatory behavior—and of the resulting implications for firm’s pricing policies—certainly merits further research. Along these same lines, so does the broader area of embedding consumer emotional rationality into the firms’ own operational decisions.

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References


**Appendix A: Proofs**

**Proof of Proposition 1**: By Lemma 1, anticipating to buy at full price, or \((p_h, 1)\), is not optimal for consumers. It remains to compare \(W(p_l, 0) = W(p_h, 0)\) and \(W(p_l, 1)\) to characterize the optimal anticipation level. We have

\[
W(p_l, 0) = W(p_h, 0) \geq W(p_l, 1)
\]

or, equivalently,

\[
q(v - p_l) \geq q(k + 1)(v - p_l) + (1 - q) \max\{v - p_h - \lambda(p_h - p_l), -\lambda v\};
\]

hence

\[
-(1 - q) \max\{v - p_h - \lambda(p_h - p_l), -\lambda v\} \geq qk(v - p_l),
\]
from which it follows that
\[ k \leq \frac{1 - q - \max\{v - p_h - \lambda(p_h - p_l), -\lambda v\}}{q} \]
\[ \leq \frac{1 - q - \min\{-v + p_h + \lambda(p_h - p_l), \lambda v\}}{q} \]
\[ \leq \frac{1 - q - \min\{(1 + \lambda)p_h - v, v - p_l + \lambda, \frac{v}{v - p_l}\}}{q}. \]

This establishes that, if \( k \) is sufficiently small, then anticipating \((p_h, 0)\) or \((p_h, 0)\) is optimal. Further, because \( v - p_l \geq 0 \) and \( v - p_h - \lambda(p_h - p_l) \leq 0 \), from (4) or (5) it follows that the customer buys only if the low price is realized.

However, if \( k \) is sufficiently large then \((p_l, 1)\) is the optimal anticipation level. By (3), the customer buys in either state if and only if \( v - p_l \geq -\lambda v \) and \( v - p_h - \lambda(p_h - p_l) \geq -\lambda v \). The first inequality holds. For the second to hold, it must be that \( v > p_h - p_l \) and \( \lambda \geq \frac{v - p_h - v}{v - p_l} \). This completes the proof. \(\Box\)

**Proof of Corollary 1:** For the customer always to consume under the anticipation level \((p_l, 1)\), we must have \( v - p_h - \lambda(p_h - p_l) \geq -\lambda v \) or, equivalently, \( p_h - v \leq \frac{1}{1 + \lambda} p_l \) which then implies \( W(p_l, 1) = qk(v - p_l) + q(v - p_l) + (1 - q)(v - p_h - \lambda(p_h - p_l)) \). Furthermore, it follows from Lemma 1 that \((p_h, 1)\) cannot be an optimal anticipation level. It remains to show that \( W(p_l, 0) = W(p_h, 0) = q(v - p_l) \leq W(p_l, 1) \). Simple algebra confirms the result. \(\Box\)

**Proof of Proposition 2:** (i) By Assumption 1, \( p_h - v > \frac{1}{1 + \lambda}(v - p_l) \); and by Corollary 1, we have \( p_h - v \leq \frac{(k + \lambda)q - \lambda}{(1 - q)(1 + \lambda)}(v - p_l) \). Therefore, it must be that \( \frac{\lambda}{q} < (k - 1) \). Because \( 0 < q < 1 \), that inequality holds only if \( \lambda < k - 1 \) or \( k > \lambda + 1 \).

(ii) The pricing policy that consists of \( q = \sqrt{\frac{\lambda}{1 + \lambda}} \), \( p_l = (1 - \frac{\lambda}{1 - q})v \), and \( p_h = \frac{\lambda}{1 + \lambda}p_l + v \) satisfies Assumption 1 and induces customers to anticipate \((p_l, 1)\) and to always consume. \(\Box\)

**Proof of Proposition 3:** The critical points solve the system of equations

\[ 1 - q = \alpha_1 + \alpha_2, \quad (A.1) \]
\[ q = -\frac{\lambda}{1 + \lambda} \alpha_1 + \frac{(k + \lambda)q - \lambda}{(1 - q)(1 + \lambda)} \alpha_2, \quad (A.2) \]
\[ p_l - p_h = -\frac{k(v - p_l)}{(1 - q)^2(1 + \lambda)} \alpha_2, \quad (A.3) \]
\[ (p_h - v - \frac{\lambda}{1 + \lambda} p_l) \alpha_1 = 0, \quad (A.4) \]
\[ (p_h - v - \frac{(k + \lambda)q - \lambda}{(1 - q)(1 + \lambda)}(v - p_l)) \alpha_2 = 0; \quad (A.5) \]

here \( \alpha_1 \) and \( \alpha_2 \) are nonnegative Lagrange multipliers associated with the constraints (10) and (11), respectively. Condition (A.1) implies that \( \alpha_1 + \alpha_2 > 0 \). By (A.2), it must be that \( q > \frac{\lambda}{1 + \lambda} \) and \( \alpha_2 > 0 \). Since \( \alpha_2 > 0 \), it follows from (A.5) that \( p_h = \frac{(k + \lambda)q - \lambda}{(1 - q)(1 + \lambda)}(v - p_l) + v \). Substituting \( p_h \) in (A.3) and then simplifying, we obtain \( \alpha_2 = \frac{(1 - q)\alpha_1}{k} \). We now substitute the value for \( \alpha_2 \) in (A.1) and simplify to obtain \( \alpha_1 = \frac{(k - 1)(1 - q)^2}{k} \).

Finally, we plug the values for \( \alpha_1 \) and \( \alpha_2 \) into (A.2) and then simplify to obtain \( q = \frac{\lambda \alpha_1 + \alpha_2}{k + \lambda} \) or \( q = \sqrt{\frac{\lambda}{1 + \lambda}} \).

Because \( \alpha_1 > 0 \), by (A.4) we have \( p_h = \frac{\lambda}{1 + \lambda} p_l + v \); this equality, together with \( p_h = \frac{(k + \lambda)q - \lambda}{(1 - q)(1 + \lambda)}(v - p_l) + v \), yields the desired result. \(\Box\)
Proof of Proposition 4: (i) The result follows because \( q^* = \sqrt{\frac{1}{k}} \). (ii) This part follows because \( p_h - p_l = \left(1 - \frac{1}{1 + \lambda} \left(1 - \sqrt{\lambda(1-\lambda)}\right)\right) \) and therefore \( \frac{d(p_h - p_l)}{d\lambda} = \frac{\sqrt{\lambda(1-\lambda)}(1+\lambda)}{2(1+\lambda)^2} \geq 0 \). The inequality follows because \( k > \lambda + 1 \).

(iii) Observe that \( \pi^* (p_l^*, p_h^*, q^*) = \frac{\lambda(\sqrt{\lambda(1-\lambda)} - \sqrt{\lambda})^2}{k(1+\lambda)\sqrt{\lambda(1-\lambda)}} v + v - c \) and so \( \frac{d\pi^*}{d\lambda} = \frac{\lambda\sqrt{\lambda(1-\lambda)}(1+\lambda)}{k^2(1+\lambda)^2} > 0 \). Moreover, \( \frac{d\pi^*}{d\lambda} = \frac{(\sqrt{\lambda(1-\lambda)} - \sqrt{\lambda})^2}{k(1+\lambda)\sqrt{\lambda(1-\lambda)}} \). Subject to \( \lambda < k - 1 \), the numerator is first positive and then negative; this implies that \( \pi^* \) first increases and then decreases in \( \lambda \). □

Proof of Proposition 5: We first identify the cases in which the customer purchases the product at either price. Observe that any random policy will require \( p_l \leq v \), since otherwise there is no element of anticipation in customer surplus.

Case 1: \( 0 \leq v - p_h \leq v - p_l \). In this case, we have
\[
W(p_h, 1) = qk(v - p_h) + q(v - p_l) + (1 - q) \max\{v - p_h - \lambda(p_h - p_l), -\lambda v\} \\
= qk(v - p_h) + q(v - p_l) + (1 - q)(v - p_h - \lambda(p_h - p_l)),
\]
\[
W(p_l, 0) = q(v - p_h) + (1 - q) \max\{v - p_h - \lambda(p_h - p_l), 0\},
\]
\[
W(p_h, 0) = q(v - p_h) + (1 - q)(v - p_h),
\]
\[
W(p_l, 1) = (1 - q)k(v - p_h) + q(v - p_l) + (1 - q)(v - p_h).
\]

Because \( W(p_h, 1) > W(p_l, 0) \) and \( W(p_h, 1) > W(p_l, 0) \), customers buy in both states irrespective of whether they anticipate \( (p_h, 1) \) or rather \( (p_l, 1) \). Two subcases are considered next.

Case 1(a): Prices are such that customers anticipate \( (p_h, 1) \). Therefore, the firm’s random sales policy must satisfy the following conditions:
\[
p_h - v \leq 0,
\]
\[
p_h - p \geq \frac{q}{1-q} (p - p_h),
\]
\[
k(q(v - p_h) - (1 - q)(v - p_h)) \geq \lambda(1 - q)(p_h - p_l),
\]
\[
qk(v - p_h) + q(v - p_l) + (1 - q)(v - p_h - \lambda(p_h - p_l)) \geq (k + 1)(v - p).
\]

The second constraint ensures that the randomizing firm’s profit per customer is no less than the uniform pricing firm’s. The third constraint is a simplification of \( W(p_h, 1) \geq W(p_h, 1) \). The fourth constraint guarantees that customers will choose the randomizing firm because it gives them a higher surplus.

Observe that the second constraint implies \( p \leq p_h \) and that the fourth constraint can be written as
\[
(k + 1)p \geq kv - qk(v - p_h) + q \lambda(1 - q)(p_h - p_l).
\]

Since \( qp_l + (1 - q)p_h \geq p \) (by the second constraint), it follows that a necessary condition for (A.6) to hold is \( qk(v - p_h) \geq k(v - p) + \lambda(1 - q)(p_h - p_l) \)—which is tighter than the third constraint. Therefore, Case 1(a) simplifies to finding \( (p_h, p_l, q) \) such that
\[
p_h - v \leq 0,
\]
\[
p_h - p \geq \frac{q}{1-q} (p - p_h),
\]
\[
k(q(v - p_h) - (1 - q)(v - p_h)) \geq \lambda(1 - q)(p_h - p_l),
\]
\[
qk(v - p_h) + q(v - p_l) + (1 - q)(v - p_h - \lambda(p_h - p_l)) \geq (k + 1)(v - p).
\]
**Case 1(b):** Prices are such that customers anticipate \((p_h, 1)\). In this case, the randomizing firm’s policy must satisfy the following constraints:

\[
\begin{align*}
& p_h - v \leq 0, \\
& p_h - p \geq \frac{q}{1 - q} (p - p_t), \\
& k(1 - q)(v - p_t) - (1 - q)(v - p_h) \leq \lambda (p_h - p_t), \\
& (1 - q)k(v - p_h) + q(v - p_t) + (1 - q)(v - p_h) \geq (k + 1)(v - p).
\end{align*}
\]

The fourth constraint simplifies to \((k + 1)p \geq kv - (1 - q)k(v - p_h) + q p_t + (1 - q)p_h\), which holds if \((1 - q)(v - p_h) \geq (v - p)\) (because \(q p_t + (1 - q)(v - p_h) \geq p\)). Since \(p \leq p_h\) (by the second constraint), we conclude that this case cannot yield a random-price policy that outperforms the uniform-price policy.

As a caveat, note that \((1 - q)(v - p_h) \geq (v - p)\) if \(v = p_h = p\); by the second constraint, that condition implies \(p = p_h = p = v\), which of course is not a random policy.

**Case 2:** \(-\lambda v \leq v - p_h \leq 0\). Here we have

\[
\begin{align*}
W(p_h, 1) &= q k(v - p_t) + q(v - p_t) + (1 - q) \max\{v - p_h - \lambda (p_h - p_t), -\lambda v\}, \\
W(p_h, 0) &= q(v - p_t) + (1 - q) \max\{v - p_h - \lambda (p_h - p_t), 0\} = q(v - p_t), \\
W(p_h, 1) &= q(v - p_t) + (1 - q)(v - p_h).
\end{align*}
\]

Observe that \(W(p_h, 1) < W(p_h, 0)\). For customers to purchase from the randomizing firm regardless of the price (i.e., in either state), the following constraints must be satisfied:

\[
\begin{align*}
& p_h - v \geq 0, \\
& p_h - v \leq \frac{\lambda}{1 + \lambda} p_t, \\
& p_h - v \leq \frac{(k + \lambda)q - \lambda}{(1 - q)(1 + \lambda)} (v - p_t), \\
& p_h - p \geq \frac{q}{1 - q} (p - p_t), \\
& q k(v - p_t) + q(v - p_t) + (1 - q)(v - p_h - \lambda (p_h - p_t)) \geq (k + 1)(v - p).
\end{align*}
\]

The second and third constraints ensure that customers prefer anticipating \((p_t, 1)\), and then consuming in either state to all other anticipation and consumption plans. The fourth constraint is that the profit of the randomizing firm must be at least as high as the profit of the uniform pricing firm. The last constraint is the incentive compatibility constraint for customers to choose the randomizing firm over the uniform pricing firm.

We conclude that only Case 1(a) and Case 2 are likely to yield random policies capable of outperforming the uniform-price policy. We shall now examine these two cases in more detail.

**Optimal pricing policy in Case 1(a):** The firm’s objective is to design a sales policy that outperforms the uniform price policy \(c < p \leq v\)—that is, \(q p_t + (1 - q) p_h - c \geq p - c\) subject to the constraints already specified. The critical points solve the following Karush-Kuhn-Tucker (KKT) conditions:

\[
\begin{align*}
p_h - p_t &= (p_t + p_h) \alpha_2 + (-k(v - p_t) + p_t - p_h - \lambda(p_h - p_t)) \alpha_3,
\end{align*}
\]

(A.7)
\[ q = -q\alpha_2 + (qk + q - \lambda(1 - q))\alpha_3, \quad \text{(A.8)} \]
\[ 1 - q = \alpha_1 - (1 - q)\alpha_2 + (1 - q + \lambda(1 - q))\alpha_3, \quad \text{(A.9)} \]
\[ (p_h - v)\alpha_1 = 0, \quad \text{(A.10)} \]
\[ (p - q \alpha_1 - (1 - q)p_h)\alpha_2 = 0, \quad \text{(A.11)} \]
\[ [(k + 1)(v - p) - (qk(v - p) + q(v - p) + (1 - q)(v - p_h - \lambda(p_h - p)))\alpha_3 = 0; \quad \text{(A.12)} \]

Here \(\alpha_1, \alpha_2,\) and \(\alpha_3\) are the Lagrange multipliers associated with Case 1(a) constraints. We simplify the first three KKT conditions as

\[ (p_h - p_i)(1 + \alpha_2 - (1 + \lambda)\alpha_3) = k(v - p_i)\alpha_3, \quad \text{(A.13)} \]
\[ (1 + \alpha_2)q = \alpha_3(q(1 + k) - \lambda(1 - q)), \quad \text{(A.14)} \]
\[ (1 - q)(1 + \alpha_2) = \alpha_1 + (1 - q)(1 + \lambda)\alpha_3, \quad \text{(A.15)} \]

respectively. We proceed to solve the system of KKT conditions and thereby characterize the critical points.

Suppose \(\alpha_1 > 0;\) it then follows from (A.10) that \(p_h = v.\) Obviously, \(p_i = v\) implies \(p_h = p_i = v,\) which is not a random policy. We can therefore use that \(p_i < v\) to simplify (A.13) and (A.14) as follows:

\[ 1 + \alpha_2 - (1 + \lambda)\alpha_3 = k\alpha_3, \quad \text{(A.16)} \]
\[ (1 + \alpha_2)q = \alpha_3(q(1 + k) - \lambda(1 - q)). \quad \text{(A.17)} \]

By (A.16) we have that \(\alpha_3 > 0\) and \(\alpha_2 = (1 + \lambda)\alpha_3 + k\alpha_3 - 1.\) Substituting \(\alpha_2\) into (A.17) yields \((1 + \lambda + k)q = (1 + k)q - \lambda(1 - q)—a contradiction.\)

Therefore, it must be that \(\alpha_1 = 0\) and \(p_h < v.\) Hence (A.13)–(A.15) can be simplified as

\[ (p_h - p_i)(1 + \alpha_2 - (1 + \lambda)\alpha_3) = k(v - p_i)\alpha_3, \quad \text{(A.18)} \]
\[ (1 + \alpha_2)q = \alpha_3(q(1 + k) - \lambda(1 - q)), \quad \text{(A.19)} \]
\[ (1 - q)(1 + \alpha_2) = (1 - q)(1 + \lambda)\alpha_3. \quad \text{(A.20)} \]

From (A.20) it follows that \(1 + \alpha_2 = (1 + \lambda)\alpha_3,\) which implies \(\alpha_3 > 0.\) We substitute \(1 + \alpha_2 = (1 + \lambda)\alpha_3\) into (A.18) and obtain \(0 = k(v - p_i)\alpha_3.\) Because \(\alpha_3 > 0,\) we must have \(v = p_i—yet this is not possible because \(p_i < p_h < v.\)

In short, Case 1(a) cannot yield a random-price policy that outperforms the uniform-price policy \(c < p \leq v.\)

**Optimal pricing policy in Case 2:** Once again, the firm’s objective is to design a sales policy that outperforms the uniform-price policy \(c < p \leq v.\) The critical points solve the following KKT conditions:

\[ p_i - p_h = -\frac{k(v - p_i)}{(1 - q)^2(1 + \lambda)}\alpha_3 - (k + 1)(v - p_i)\alpha_4 + (v - p_h - \lambda(p_h - p_i))\alpha_4 + (p_h - p_i)\alpha_5, \quad \text{(A.21)} \]
\[ q = -\frac{\lambda}{1 + \lambda}\alpha_2 + \frac{(k + \lambda)q - \lambda}{(1 - q)(1 + \lambda)}\alpha_3 + (k + 1)qa_4 - \lambda(1 - q)\alpha_4 - qa_5, \quad \text{(A.22)} \]
\[ 1 - q = -\alpha_1 + \alpha_2 + \alpha_3 + (1 + \lambda)(1 - q)\alpha_4 - (1 - q)\alpha_5, \quad \text{(A.23)} \]
\[ \alpha_1(v - p_h) = 0, \quad \text{(A.24)} \]
\[ \alpha_2(p_h - v - \frac{\lambda}{1 + \lambda}p_i) = 0, \quad \text{(A.25)} \]
Assume first that $\alpha > 0$. From (A.24) it necessarily follows that $p_h = v$. Substituting into (A.21), we obtain
\[ p_t - v = \left( \frac{k}{(1 - q)(1 + \lambda)} \alpha_3 + (k + 1 + \lambda) \alpha_4 - \alpha_5 \right) (p_t - v). \] If $p_t = v$, then $p_t = p_h = v$ and there is no random policy. Otherwise, (A.21)–(A.23) simplify as (respectively)
\[ 1 = \frac{k}{(1 - q)(1 + \lambda)} \alpha_3 + (k + 1 + \lambda) \alpha_4 - \alpha_5, \]
\[ q = -\frac{\lambda}{1 + \lambda} \alpha_2 + \frac{(k + \lambda)q - \lambda}{(1 - q)(1 + \lambda)} \alpha_3 + q(k + 1) \alpha_4 - \lambda(1 - q) \alpha_4 - \alpha_5, \]
\[ (1 - q)(1 + \alpha_5 - (1 + \lambda) \alpha_4) = -\alpha_1 + \alpha_2 + \alpha_3. \]

From (A.29) we obtain, $1 - (k + \lambda + 1) \alpha_4 + \alpha_5 = -\frac{k}{(1 - q)(1 + \lambda)} \alpha_3$, and (A.30) leads to $q - (1 - (k + \lambda + 1) \alpha_4 + \alpha_5) = -\frac{\lambda}{1 + \lambda} \alpha_2 + \frac{(k + \lambda)q - \lambda}{(1 - q)(1 + \lambda)} \alpha_3 - \lambda \alpha_4$. These two equalities together imply that $\alpha_3 > 0$ because otherwise $\alpha_2 = \alpha_3 = \alpha_4 = 0$ and so, by (A.31), $(1 - q)(1 + \alpha_5 - (1 + \lambda) \alpha_4) = -\alpha_1 + \alpha_2 + \alpha_3$.

Because $\alpha_3 > 0$, by (A.26) we have $p_h = v + \frac{(k + \lambda)q - \lambda}{(1 - q)(1 + \lambda)} (p_t - v)$. Since $p_h = v$, it follows that $\frac{(k + \lambda)q - \lambda}{(1 - q)(1 + \lambda)} (p_t - v) = 0$, which implies that either $p_t = v$ or $q = \frac{\lambda}{1 + \lambda}$. The former means $p_h = p_t = v$ and hence no random policy, so the latter must hold.

Substituting $q = \frac{\lambda}{1 + \lambda}$ into (A.30), we obtain $\frac{1}{1 + \lambda} (1 - (k + \lambda + 1) \alpha_4 + \alpha_5) = -\frac{\lambda}{1 + \lambda} \alpha_2 - \lambda \alpha_4$. The right-hand side (RHS) of this equation is nonpositive and so $1 - (k + \lambda + 1) \alpha_4 + \alpha_5 \leq 0$, which together with (A.29) implies $\frac{k}{(1 - q)(1 + \lambda)} \alpha_3 \leq 0$—a contradiction. We conclude that $\alpha_1 = 0$ and hence, by (A.24), that $p_h \geq v$.

To continue, we rewrite equations (A.21)–(A.23) as follows:
\[ (p_t - p_h)(1 - \lambda \alpha_4 + \alpha_5) = -\frac{k}{(1 - q)(1 + \lambda)} (v - p_t) \alpha_3 - (k + 1)(v - p_t) \alpha_4 + (v - p_h) \alpha_4, \]
\[ q(1 - \lambda \alpha_4 + \alpha_5) = -\frac{\lambda}{1 + \lambda} \alpha_2 + \frac{(k + \lambda)q - \lambda}{(1 - q)(1 + \lambda)} \alpha_3 + q(k + 1) \alpha_4 - \lambda \alpha_4, \]
\[ (1 - q)(1 + \alpha_5) = \alpha_2 + \alpha_3 + (1 + \lambda) \alpha_4. \]

Substituting $1 + \alpha_5$ from (A.34) into (A.32) and (A.33) then yields
\[ (p_t - p_h) \frac{\alpha_2 + \alpha_3 + (1 - q) \alpha_4}{1 - q} = -\frac{k}{(1 - q)(1 + \lambda)} (v - p_t) \alpha_3 - (k + 1)(v - p_t) \alpha_4 + (v - p_h) \alpha_4, \]
\[ \frac{\alpha_2 + \alpha_3 + (1 - q) \alpha_4}{1 - q} = -\frac{\lambda}{1 + \lambda} \alpha_2 + \frac{(k + \lambda)q - \lambda}{(1 - q)(1 + \lambda)} \alpha_3 + q(k + 1) \alpha_4 - \lambda \alpha_4. \]

We now consider two cases.

First suppose $\alpha_4 = 0$, which implies that the customer incentive compatibility constraint is not binding. Then (A.35) and (A.36) simplify as
\[ (p_t - p_h) \alpha_3 = -\frac{k}{(1 - q)(1 + \lambda)} (v - p_t) \alpha_3, \]
\[ q(1 + \lambda) \alpha_3 = -\lambda (1 - q) \alpha_2 + ((k + \lambda)q - \lambda) \alpha_3. \]

If $\alpha_3 = 0$ then by (A.35), $\alpha_2 = 0$. Because $\alpha_4 = \alpha_3 = \alpha_2 = 0$, by (A.34) we have $(1 - q)(1 + \alpha_5) = 0$—a contradiction. Hence it must be that $\alpha_3 > 0$, which implies $p_h = v + \frac{(k + \lambda)q - \lambda}{(1 - q)(1 + \lambda)} (v - p_t)$ (see (A.26)).
Suppose now that \( \alpha_2 = 0 \). From (A.37) we have \( (p_t - p_h)\alpha_3 = -\frac{\alpha}{(1-q)(1+\lambda)}(v - p_t)\alpha_3 \), which together with the equality \( p_h = v + \frac{(k+\lambda)\alpha_3}{(1-q)(1+\lambda)}(v - p_t) \) implies \( k = 1 \). Substituting this value for \( k \) into (A.38) yields

\[
q(1+\lambda)\alpha_3 = ((1+\lambda)q - \lambda)\alpha_3, 
\]

a contradiction because \( \lambda > 0 \). Hence we must have that \( \alpha_2 > 0 \) and so, by (A.25),

\[
p_h = v + \frac{\alpha}{1+\lambda}p_t.
\]

Substituting \( p_h = v + \frac{(k+\lambda)\alpha_3}{(1-q)(1+\lambda)}(v - p_1) \) into (A.37) and recalling that \( \alpha_2 > 0 \), we obtain

\[
(kq - q + 1)\alpha_2 = (1-q)(k-1)\alpha_3,
\]

which in turn implies \( k > 1 \) and hence that \( \alpha_3 = \frac{kq-q+1}{(1-q)(k-1)}\alpha_2 \). Substituting \( \alpha_3 \) into (A.38) and then solving for \( q \), we derive

\[
q = \sqrt{\frac{k}{\lambda}} \text{ if } k > \lambda + 1 \quad \text{(otherwise, there is no solution for } q \text{ and so this case becomes void)}.
\]

We are now in a position to solve \( p_h = v + \frac{\alpha}{1+\lambda}p_t \) and \( p_h = v + \frac{(k+\lambda)\alpha_3}{(1-q)(1+\lambda)}(v - p_1) \) for \( p_t \) and \( p_h \), which yields

\[
p_t = (1 - \frac{\alpha}{k}(1-q))v \quad \text{and} \quad p_h = v + \frac{\alpha}{1+\lambda}p_t.
\]

For these values of \( p_t \), \( p_h \), and \( q \), we have

\[
qp_h + (1-q)p_t > p
\]

(see Proposition 3); hence \( \alpha_5 = 0 \). Substituting \( \alpha_5 \) into (A.34), we get

\[
\alpha_2 = \frac{(1-q)^2(k-1)}{k}.
\]

The value for \( \alpha_3 \) is computed from the equation

\[
\alpha_3 = \frac{kq-q+1}{(1-q)(k-1)}\alpha_2.
\]

Finally, since \( \alpha_4 = 0 \), it follows from (A.27) that the values for \( p_t \), \( p_h \), and \( q \) are such that

\[
(k+1)(v - p) - q(k+1)(v - p_t) - (1-q)(v - p_h - \lambda(p_h - p_t)) \leq 0
\]

or, equivalently,

\[
p \geq \left(1 - \frac{\lambda(1 - \sqrt{\frac{k}{\lambda}})}{k(k+1)}\right)v. \tag{A.39}
\]

To summarize, we found a series of variables \( p_t \), \( p_h \), \( q \), and \( \alpha_i \) \( (i = 1, \ldots, 5) \) that, subject to \( k > \lambda + 1 \) and (A.39), describe a random policy under which the randomizing firm outperforms the uniform pricing firm. Furthermore, customers strictly prefer the former to the latter because the incentive compatibility constraint is strict. We will refer to this policy as **Policy 1**.

Next we analyze the case where \( \alpha_4 > 0 \). From (A.27) we derive that

\[
q(k+1)(v - p_t) + (1-q)(v - p_h - \lambda(p_h - p_t)) = (k+1)(v - p). \tag{A.40}
\]

Suppose now \( \alpha_3 = 0 \). Then (A.35) and (A.36) simplify as

\[
(p_t - p_h)\alpha_2 + \frac{(1-q)\alpha_4}{1-q} = -(k+1)(v - p_t)\alpha_4 + (v - p_h)\alpha_4, \tag{A.41}
\]

\[
\frac{\alpha_2 + (1-q)\alpha_4}{1-q} = -\frac{\lambda}{1+\lambda}\alpha_2 + q(1+k)\alpha_4 - \lambda\alpha_4. \tag{A.42}
\]

It is obvious that \( \alpha_2 > 0 \) because otherwise (A.41) would be equivalent to \(-k(v - p_t) = 0\) (i.e., \( p_t = v \)), which is not admissible. The reason is that, by (A.40) and \( p_t = v \), we have \( (1-q)(1+\lambda)(v - p_h) = (k+1)(v - p) \); the left-hand side (LHS) is nonpositive because \( p_h \geq v \) while the RHS is nonnegative because \( p \leq v \) (the equality would hold only if \( p_t = p_h = v = p \), which again is not a random policy).

Because \( \alpha_2 > 0 \), from (A.25) we obtain \( p_h = \frac{\lambda}{1+\lambda}p_t + v \). Substituting this term into (A.41), we obtain

\[
(p_t - v)\left(\frac{\alpha_2}{1-q} - k\alpha_4\right) = \frac{\lambda}{1+\lambda}\frac{\alpha_2}{1-q}p_t. \tag{A.43}
\]

Since the LHS of (A.43) is positive, it follows that \( \frac{\alpha_2}{1-q} < k\alpha_4 \).

Assume now that \( \alpha_5 = 0 \). By (A.34), either \( 1 - q = \alpha_2 + (1+\lambda)(1-q)\alpha_4 \) or \( \alpha_2 = (1-q)(1+\lambda)\alpha_4 \). We note two inequalities: (i) because \( \frac{\alpha_2}{1-q} < k\alpha_4 \), it must be that \( (1-q)(1+\lambda)\alpha_4 < (1-q)k\alpha_4 \) or,
equivalently, $\alpha_4 > \frac{1}{1+\lambda/\varepsilon}$; (ii) $\alpha_2 > 0$, so we must have either $1 - (1 + \lambda)\alpha_4 > 0$ or $\alpha_4 < \frac{1}{1+\lambda}$. Combining (i) and (ii) results in $\frac{1}{1+\lambda/\varepsilon} < \alpha_4 < \frac{1}{1+\lambda}$, and substituting $\alpha_2$ into (A.43) yields $p_1 = \frac{(k+\lambda+1)\alpha_4 - 1}{(k+\lambda+k+\lambda+1)\alpha_4 - 1}(1+\lambda)v$.

Note that $p_1 > 0$ because $(1 + \lambda + k)\alpha_4 - 1 > 0$; in addition, $p_1 < v$ because $v - p_1 = \frac{(\lambda(1-\alpha_2(1+\lambda))/((k+\lambda+k+\lambda+1)\alpha_4 - 1) > 0$. The latter inequality holds because $1 - (1 + \lambda)\alpha_4 > 0$.

We now substitute $\alpha_4$ into (A.38) and solve for $q$, which yields $q = \frac{\lambda}{(k+\lambda+k+\lambda+1)\alpha_4 - 1} > 0$ For $q < 1$, we must have $\alpha_4 > \frac{1}{1+\lambda}$. Combining this inequality with (i) and (ii) from before yields $\frac{1}{1+\lambda} < \alpha_4 < \frac{1}{1+\lambda}$, a necessary condition for which is $k > \lambda$.

To compute $\alpha_4$, we substitute $\alpha_2$, $p_1$, $p_b$, and $q$ into (A.40) to obtain the following quadratic equation in $\alpha_4$:

$$-(1+k)^2(1+\lambda)^2\alpha_4^2 - 2(k\lambda + k + \lambda + 1)\alpha_4 - k\lambda + \lambda(v = (k+1)(v-p)).$$

The roots for $\alpha_4$ are

$$\alpha_4 = \frac{A \pm \sqrt{A^2 - 4B}}{2B},$$

where $A = (k+1)(v-p) + \lambda(v) > 0$. It is easy to show that $\alpha_4 = \frac{1-\sqrt{A^2 - 4B}}{2B} < \frac{1}{1+\lambda}$. Therefore, only $\alpha_4 = \frac{A + \sqrt{A^2 - 4B}}{2B}$ is admissible. We next need to check that $\frac{1}{1+\lambda} < \alpha_4 < \frac{1}{1+\lambda}$.

We have $\alpha_4 - \frac{1}{1+\lambda} = \frac{\lambda}{(k+1)(1+\lambda)}$. It follows that $\alpha_4 > \frac{1}{1+\lambda}$ if $p > \frac{2}{1+\lambda}v$. Moreover, $\frac{1}{1+\lambda} < \alpha_4 = \frac{\lambda}{(k+1)(1+\lambda)}$.

For $\alpha_4 < \frac{1}{1+\lambda}$, it must be that $p < \frac{k^2 + k\lambda - \lambda^2}{v(k+1)}$, which holds because $p < \frac{k^2 + k\lambda - \lambda^2}{v(k+1)}$.

Finally, we verify that—for the variable values we found—the inequalities $q(1-q)p_1 + p \geq p$ and $p_b \leq v + \frac{(k+\lambda)q-\lambda}{(1-\alpha_2)(1+\lambda)}(v-p_1)$ both hold. We have

$$q(1-q)p_1 + p = \frac{A + kv - 2\sqrt{k\lambda v}}{(1+\lambda)(1+k)} < 0 \quad \text{(A.44)}$$

and also

$$v + \frac{(k+\lambda)q - \lambda}{(1-q)(1+\lambda)}(v-p_1) - p = \frac{A(k^2(1+k)(v-p) - k\lambda v + \lambda \sqrt{k\lambda v})}{(1+\lambda)(1+k)\sqrt{kvA}}(-A + \sqrt{kvA}).$$

Given that $-A + \sqrt{kvA} > 0$, we need only check that $B = k^2(1+k)(v-p) - k\lambda v + \lambda \sqrt{k\lambda v} > 0$. We have

$$\frac{\partial B}{\partial p} = -k^3 - k^2 - \frac{k\lambda v(k+1)}{2\sqrt{k\lambda v}} < 0, B(p = \frac{1+\lambda}{1+\lambda}) = k^2v(k-\lambda) > 0, \text{ and } B(p = v) = \lambda \sqrt{k\lambda - \lambda}v < 0.$$

Therefore, $B = 0$ has a unique solution in $p \in (1+\lambda, v]$. This solution is

$$p = \frac{2k^3 + 2k^3 - 2k \lambda \lambda + \lambda \sqrt{(4k^3 + 4k^2 + \lambda v)}(k+1)}{2k^3(k+1)}.$$

So for $p_b \leq v + \frac{(k+\lambda)q-\lambda}{(1-q)(1+\lambda)}(v-p_1)$ to hold, it must be that

$$1 + \frac{\lambda}{1+k} < p \leq \frac{2k^3 + 2k^3 - 2k \lambda \lambda + \lambda \sqrt{(4k^3 + 4k^2 + \lambda v)}(k+1)}{2k^3(k+1)}.$$

In summary, we have identified a random-price policy under which $\alpha_1 = \alpha_3 = \alpha_5 = 0$, $\alpha_2 = (1-q)(1-(1 + \lambda)\alpha_4)$, $\alpha_2 = \frac{A + \sqrt{A^2 - 4B}}{2B}$. A $q = \frac{\lambda}{(k+\lambda+k+\lambda+1)\alpha_4 - 1}$, $p_1 = \frac{(k+\lambda+1)\alpha_4 - 1}{(k+\lambda+k+\lambda+1)\alpha_4 - 1}(1+\lambda)v$, and $p_b = v + \frac{\lambda}{1+k}p_1$ subject to the conditions that $k > \lambda$ and $\frac{1+\lambda}{1+k} < v \leq \frac{2k^3 + 2k^3 - 2k \lambda \lambda + \lambda \sqrt{(4k^3 + 4k^2 + \lambda v)}(k+1)}{2k^3(k+1)}$. This policy ensures that customers obtain the same surplus under the randomized and uniform pricing policies but that the randomizing firm is strictly better-off. We will refer to this policy as **Policy 2**.
We next address the case $\alpha > 0$ which implies $qp + (1 - q)p = p$. In this case the incentive compatibility constraint is binding and so customers are indifferent between the two firms; furthermore, profits for the randomizing firm are the same as for the firm that adopts the uniform pricing policy. We are not interested in this case because there is no point in randomizing if it benefits neither the firm nor the customers.\footnote{It is easy to see that there is no random policy satisfying the constraints. Substitute $p_h = v + \frac{\lambda}{1 + \lambda} p_t$ into $qp + (1 - q)p = p$ and obtain $p_t = \frac{(\lambda + qv - \lambda q)p_t}{\lambda + (1 + \lambda) p_t}$; then substitute $p_t$ into (A.41) and obtain $\alpha_2 = \frac{\lambda qv + (1 + \lambda) q - 2\lambda q}{k(1 + \lambda) p_t}$. Finally, substitute $\alpha_2$ into (A.38), which yields

$$
\frac{k\lambda qv + k\lambda p - 2k\lambda v + kp - kv - \lambda v + p - v}{-(\lambda v - p + v)} \alpha_4 q = -k\lambda^2 q(4v - pv) + \lambda(k\lambda + \lambda p - 2\lambda q)(2p - 3v) + \lambda^2 v(\lambda - q) + (k + 1)(\lambda - q)(v - p) \frac{\alpha_4 q}{(1 + \lambda)(\lambda v - p + v)}.
$$

Yet, this equation holds only if $\alpha_4 = 0$—a contradiction.}

We now investigate the case where $\alpha_3 > 0$. First we observe that $\alpha_5 = 0$ under the random sales policy (if one exists), since otherwise customers are indifferent between the two firms and the randomizing firm is no better-off than the uniform pricing firm. Therefore, we have the following set of equations:

\[
\begin{align*}
1 - q &= \alpha_2 + \alpha_3 + (1 + \lambda)(1 - q)\alpha_4, \\
q(1 - \lambda\alpha_4) &= -\frac{\lambda}{1 + \lambda} \alpha_2 + \frac{(k + \lambda)q - \lambda}{(1 - q)(1 + \lambda)} \alpha_3 + q(1 + k)\alpha_4 - \lambda\alpha_4, \\
(p_t - p_h) (1 - \lambda\alpha_4) &= \frac{k}{(1 - q)^2(1 + \lambda)} (v - p_t) \alpha_3 - (k + 1)(v - p_t)\alpha_4 + (v - p_h)\alpha_4, \\
qu = \frac{(k + \lambda)q - \lambda}{(1 - q)(1 + \lambda)} (v - p_t).
\end{align*}
\]

We substitute (A.49) into (A.48) and obtain $q(v - p_t) = (k + 1)(v - p)$ or

\[
p_t = v - \frac{k + 1}{q} (v - p);
\]

hence, by (A.49),

\[
p_h = v + \frac{(k + 1)((k + \lambda)q - \lambda)}{q(1 - q)(1 + \lambda)} (v - p).
\]

We now substitute $p_h$ into (A.47) and then simplify which yields

\[
1 - q + kq = \frac{k}{1 - q} \alpha_3 + (1 + \lambda)(k + 1 - q)\alpha_4.
\]

Suppose that $\alpha_2 = 0$. From (A.45) we have $\alpha_3 = (1 - q)(1 - (1 + \lambda)\alpha_4)$, which implies that $\alpha_4 < \frac{1}{1 + \lambda}$.

Substituting for $\alpha_3$ in (A.52) and simplifying, we obtain $(1 - k)(1 - q) = (1 - q)(1 + \lambda)\alpha_4$ or $\alpha_4 = \frac{k}{k + 1}$, hence $\alpha_3 = k(1 - q)$, where necessarily $k < 1$. If we substitute the values for $\alpha_3$ and $\alpha_4$ into (A.46), the result is

$q = \frac{\lambda^2 - \lambda + 2}{1 + \lambda}$ or $\lambda = 0$—a contradiction. It must therefore hold that $\alpha_2 > 0$, which implies $p_h = v + \frac{\lambda}{1 + \lambda} p_t$.
Substituting the value for \( q \) into (A.50) and (A.51) now yields \( p_l = v - \frac{(k+1)\lambda v}{\lambda v - k(k+1)(v-p)}(v-p) \) and \( p_h = v + \frac{k-\lambda}{\lambda v - k(k+1)(v-p)}(v-p) \). By (A.53), \( \lambda v - k(k+1)(v-p) > 0 \). Moreover, \( k(k+\lambda+1)(v-p) < \lambda p \) (because \( p_h \geq v \)) or, equivalently, \( p > \frac{k(k+\lambda+1)}{k+\lambda+1} v \). \( \text{(A.54)} \)

We next verify that \( q p_l + (1-q)p_h > p \), or
\[
q p_l + (1-q)p_h - p = \frac{k(v-p) \left(-\left(k^3+k^2+\lambda\right)(v-p)+\lambda (k p - \lambda v)\right)}{(-k(k+1)(v-p)+\lambda v)(1+\lambda)} > 0.
\] \( \text{(A.55)} \)

Equation (A.55) holds if \( (k^3+k^2+\lambda)(v-p) < \lambda (k p - \lambda v) \), which implies that \( k p - \lambda v > 0 \) or \( p > \frac{\lambda}{k} v \) (which itself requires \( k > \lambda \)). Furthermore, we must have \( p > \frac{k^2(k+1) + \lambda(\lambda+1)}{(k+1)(k^2+\lambda)} v \). \( \text{(A.56)} \)

A comparison of (A.53), (A.54), and (A.56) in light of the inequality \( p > \frac{\lambda}{k} v \) reveals that (A.56) is the tightest bound on \( p \).

So far we have derived the values for \( p_l \) and \( p_h \) and \( q \) (in terms of the parameters), a bound on \( p \), and the inequality \( k \geq \lambda \). Now we shall determine the values for \( \alpha_2 \), \( \alpha_3 \), and \( \alpha_4 \). If we substitute \( p_l \) and \( p_h \) into (A.45)–(A.47), then
\[
\alpha_2 = \frac{\lambda(1-q)^2}{kq^2}, \quad \alpha_3 = \frac{(1-q)(k^2q^2-k\lambda+q\lambda-\lambda)}{kq^2}, \quad \alpha_4 = \frac{q^2(1-k)+\lambda}{q^2(1+\lambda)}.
\]

Obviously, \( \alpha_2 > 0 \). For \( \alpha_3, \alpha_4 > 0 \), it must be that both (i) \( k^2q^2 - \lambda(k-q+1) > 0 \) and (ii) \( q^2(1-k) + \lambda > 0 \). We substitute \( q \) into (i), which then simplifies to
\[
\frac{k^4(k^2+2k+1)p^2}{\lambda^2 v^2} = \frac{k(k+1)(2k^4+2k^3-2k^2\lambda-\lambda^2)p}{\lambda^2 v^2} + \frac{k(k^5+2k^4-2k^3\lambda+k^3-2k^2\lambda-\lambda^2)}{\lambda^2} > 0.
\]

For the inequality to hold, it must be that
\[
p \in \left[0, \frac{(2k^4+2k^3-2k^2\lambda-\lambda^2)\sqrt{4k^4+4k^3+\lambda}}{2k^3(k+1)}v\right],
\]
or
\[
p \in \left(\frac{2k^4+2k^3-2k^2\lambda-\lambda^2+\sqrt{4k^4+4k^3+\lambda}}{2k^3(k+1)}v, v\right].
\]

Recall (A.56) and observe that
\[
\frac{2k^4+2k^3-2k^2\lambda-\lambda^2+\sqrt{4k^4+4k^3+\lambda}}{2k^3(k+1)}v < \frac{k^2(k+1)+\lambda(\lambda+1)}{(k+1)(k^2+\lambda)}v,
\]
from which it follows that the first of these two intervals for \( p \) is not admissible. So for \( \alpha_3 > 0 \) we must have
\[
p \in \left(\frac{2k^4+2k^3-2k^2\lambda-\lambda^2+\sqrt{4k^4+4k^3+\lambda}}{2k^3(k+1)}v, v\right].
\]

Now we turn to \( \alpha_4 > 0 \). First, observe that if \( k \leq 1 \) then \( \alpha_4 > 0 \). If \( k > 1 \) then \( q^2 < \frac{\lambda}{k-1} \), which, by substituting the value of \( q \), simplifies to
\[
\frac{k^2(k^3+k^2-k-1)p^2}{\lambda^2 v^2} + \frac{2k(k-1)(k+1)(k^2+k-\lambda)p}{\lambda^2 v} - \frac{k^5+4k^4-2k^3\lambda-k^3+2k^2\lambda-\lambda^2-2k^2\lambda-\lambda^2}{\lambda^2} > 0.
\]
Because \( k > 1 \) and so \( k^3+k^2-k-1 > 0 \), the inequality holds if and only if
\[
\frac{k^3-k\lambda-\lambda\sqrt{\lambda(k-1)}-k+\lambda}{(k-1)(k+1)k} < p < \frac{k^3-k\lambda+\lambda\sqrt{\lambda(k-1)}-k+\lambda}{(k-1)(1+k)k}.
\]
Recall (A.56) and that $k > \lambda$. Then can verify that
\[
\frac{k^3 - k\lambda - \lambda\sqrt{\lambda(k - 1)}}{(k - 1)(1 + k)} v < \frac{k^2(k + 1) + \lambda(\lambda + 1)}{(k + 1)(k^2 + \lambda)} v
\]
and also that
\[
\frac{k^3 - k\lambda + \lambda\sqrt{\lambda(k - 1)}}{(k - 1)(1 + k)} v > \frac{k^2(k + 1) + \lambda(\lambda + 1)}{(k + 1)(k^2 + \lambda)} v.
\]
Therefore, the LHS is positive if \(\frac{k^3(k(k+1) + \lambda(\lambda+1))}{(k+1)(k^2 + \lambda)} v < \frac{k^3 - k\lambda + \lambda\sqrt{\lambda(k - 1) - k \lambda}}{(k - 1)(1 + k)} v\). We finally note that
\[
\frac{k^3 - k\lambda + \lambda\sqrt{\lambda(k - 1) - k \lambda}}{(k - 1)(1 + k)} v \leq v \text{ if and only if } k > \lambda + 1.
\]

We summarize this case below and refer to it as **Policy 3**. Under the random sales policy whereby
\[
p_t = v - \frac{k(k+1)\lambda}{N - k(k+1)(v-p)}(v-p), \quad p_u = v + \frac{k(k+1)(p-v)}{N - k(k+1)(v-p)},\quad \lambda = 1 - \frac{k(k+1)}{N}(v-p),
\]
the randomizing firm obtains a higher profit than the uniform pricing firm and customers are indifferent between the two firms under the following conditions:

**Policy 3-1** if $\lambda < k < 1$ and $p \in \left( \frac{2k^4 + 2k^3 - 2k^2\lambda - \lambda^2 + \lambda\sqrt{\lambda(4k^3 + 4k^2 + \lambda)}}{2k^3(k+1)}, v \right]$;

**Policy 3-2** if $1 < k < \lambda + 1$, $\lambda < k$, and $p \in \left( \frac{k^2(k+1) + \lambda(\lambda+1)}{(k+1)(k^2 + \lambda)}, v \right]$;

**Policy 3-3** if $k > \lambda + 1$ and $p \in \left( \frac{k^2(k+1) + \lambda(\lambda+1)}{(k+1)(k^2 + \lambda)}, v \right], \frac{k^3 - k\lambda + \lambda\sqrt{\lambda(k - 1) - k \lambda}}{(k - 1)(1 + k)} v \right]$.

We now compare policy 1, policy 2, and policy 3 in the setting of Case 2 and, in so doing, identify which policy is optimal for the randomizing firm in response to the uniform pricing firm.

**Case (i):** $\lambda < k < 1$. In this case,
- if $p \in \left( \frac{1 + \lambda}{1+k}, \frac{2k^4 + 2k^3 - 2k^2\lambda - \lambda^2 + \lambda\sqrt{\lambda(4k^3 + 4k^2 + \lambda)}}{2k^3(k+1)} \right]$, then Policy 2 is optimal;
- if $p \in \left( \frac{2k^4 + 2k^3 - 2k^2\lambda - \lambda^2 + \lambda\sqrt{\lambda(4k^3 + 4k^2 + \lambda)}}{2k^3(k+1)}, v \right]$, then Policy 3-1 is optimal.

**Case (ii):** $\lambda < k$ and $1 < k < 1 + \lambda$. In this case, it easily follows that
- if $p \in \left( \frac{1 + \lambda}{1+k}, \frac{k^2(k+1) + \lambda(\lambda+1)}{(k+1)(k^2 + \lambda)} \right]$, then Policy 2 is optimal;
- if $p \in \left( \frac{k^2(k+1) + \lambda(\lambda+1)}{(k+1)(k^2 + \lambda)}, v \right]$, then Policy 3-2 is optimal.

To see why, first observe that
\[
\frac{k^2(k+1) + \lambda(\lambda+1)}{(k+1)(k^2 + \lambda)} v < \frac{2k^4 + 2k^3 - 2k^2\lambda - \lambda^2 + \lambda\sqrt{\lambda(4k^3 + 4k^2 + \lambda)}}{2k^3(k+1)} v.
\]
This means that, for the interval $p \in \left( \frac{1 + \lambda}{1+k}, \frac{k^2(k+1) + \lambda(\lambda+1)}{(k+1)(k^2 + \lambda)} \right]$, only Policy 2 applies; also, only Policy 3-2 applies for the interval
\[
p \in \left( \frac{2k^4 + 2k^3 - 2k^2\lambda - \lambda^2 + \lambda\sqrt{\lambda(4k^3 + 4k^2 + \lambda)}}{2k^3(k+1)}, v \right].
\]

For the interval $p \in \left( \frac{k^2(k+1) + \lambda(\lambda+1)}{(k+1)(k^2 + \lambda)}, \frac{2k^4 + 2k^3 - 2k^2\lambda - \lambda^2 + \lambda\sqrt{\lambda(4k^3 + 4k^2 + \lambda)}}{2k^3(k+1)} \right]$, Policy 2 and Policy 3-2 both apply. To identify which is preferred by the firm, we must compare the profits under each policy when $p$ is in this interval.

The profits under Policies 2 and 3-2 are given in (A.44) and (A.55), respectively. We have
\[
\pi(\text{under Policy 2}) - \pi(\text{under Policy 3-2}) = \frac{k^3(k + 1)p^2 - \left( 2k^4 + 2k^3 - 2k^2\lambda + \lambda^2 \right) v + 2k\lambda\sqrt{kvA}}{(1 + \lambda)(\lambda v - k(k + 1)(v - p))}
\]
\[
+ \frac{(k^5 + 2k^4 - 2k^3\lambda + k^2\lambda^2 - 2k^2\lambda + 2k\lambda^2 + \lambda^2 + \lambda^2) v^2 + 2\sqrt{kvA}(k^2 + k - \lambda) v}{(1 + \lambda)(\lambda v - k(k + 1)(v - p))}.
\]
This expression is quadratic in $p$ and admits two roots:
\[
p = \frac{2k^4 + 2k^3 - 2k^2\lambda - \lambda^2 \pm \lambda\sqrt{\lambda(4k^3 + 4k^2 + \lambda)}}{2k^3(k+1)} v.
\]
Straightforward algebraic manipulation yields \( \frac{2k^2+2k^3-2k^2\lambda-\lambda^2-\lambda\sqrt{4k^3+4k^2+\lambda}}{2k^3(k+1)}v < \frac{k^2(k+1)+\lambda(\lambda+1)}{(k+1)(k^2+\lambda)}v \). This means that \( \pi(\text{under Policy 2}) - \pi(\text{under Policy 3-2}) < 0 \) for all \( p \in (\frac{k^2(k+1)+\lambda(\lambda+1)}{(k+1)(k^2+\lambda)}v, \frac{2k^4+2k^3-2k^2\lambda-\lambda^2+\lambda\sqrt{4k^3+4k^2+\lambda}}{2k^3(k+1)}v) \). Therefore, Policy 3-2 is preferred by the firm to Policy 2.

**Case (iii):** \( k > \lambda + 1 \). In this case, Policies 1, 2, and 3-3 apply. We must compare these three policies to identify which is preferred by the firm. By algebra, we observe that

\[
1 + \frac{\lambda}{k} < \frac{k^2(k+1)+\lambda(\lambda+1)}{(k+1)(k^2+\lambda)}v < \frac{2k^4+2k^3-2k^2\lambda-\lambda^2+\lambda\lambda\sqrt{4k^3+4k^2+\lambda}}{2k^3(k+1)}v < \frac{k^3-k\lambda+\lambda\sqrt{4k^3+4k^2+\lambda}}{(k-1)(k+1)}v < v.
\]

We first compare Policies 2 and 3-3. Clearly, if \( \frac{1+\lambda}{k+1}v < \frac{k^2(k+1)+\lambda(\lambda+1)}{(k+1)(k^2+\lambda)}v \) then only Policy 2 applies. Likewise, if \( \frac{2k^4+2k^3-2k^2\lambda-\lambda^2+\lambda\lambda\sqrt{4k^3+4k^2+\lambda}}{2k^3(k+1)}v < \frac{k^3-k\lambda+\lambda\lambda\sqrt{4k^3+4k^2+\lambda}}{(k-1)(k+1)}v < v \) then Policy 3-3 applies. We need to compare Policies 2 and 3-3 for the interval \( \frac{k^3-k\lambda+\lambda\lambda\sqrt{4k^3+4k^2+\lambda}}{(k-1)(k+1)}v < v < \frac{2k^4+2k^3-2k^2\lambda-\lambda^2+\lambda\lambda\sqrt{4k^3+4k^2+\lambda}}{2k^3(k+1)}v \).

From Case (ii) we know that \( \pi(\text{under Policy 2}) - \pi(\text{under Policy 3-3}) < 0 \) over this interval (note that Policies 3-2 and 3-3 yield the same profit). Therefore, Policy 3-3 applies for \( \frac{k^2(k+1)+\lambda(\lambda+1)}{(k+1)(k^2+\lambda)}v < p < \frac{2k^4+2k^3-2k^2\lambda-\lambda^2+\lambda\lambda\sqrt{4k^3+4k^2+\lambda}}{2k^3(k+1)}v \).

Now we also incorporate Policy 1. From the preceding comparison, we know that, for \( \frac{1+\lambda}{k+1}v < p < \frac{k^2(k+1)+\lambda(\lambda+1)}{(k+1)(k^2+\lambda)}v \), Policy 2 yields higher profits than does Policy 3-3. Policy 1 does not apply to this range of \( p \). This is because, by algebra, \( \frac{k^2(k+1)+\lambda(\lambda+1)}{(k+1)(k^2+\lambda)} < \frac{1}{k^2+\lambda}v < \frac{k^2(k+1)+\lambda(\lambda+1)}{(k+1)(k^2+\lambda)}v \) for \( k > \lambda + 1 \).

If \( \frac{k^2(k+1)+\lambda(\lambda+1)}{(k+1)(k^2+\lambda)}v < \frac{k^3-k\lambda+\lambda\lambda\sqrt{4k^3+4k^2+\lambda}}{(k-1)(k+1)}v \) then Policy 3-3 applies. Once again, Policy 1 does not apply in this case because \( \frac{k^3-k\lambda+\lambda\lambda\sqrt{4k^3+4k^2+\lambda}}{(k-1)(k+1)}v < \frac{1-\lambda(1-\sqrt{4k^2+\lambda})}{k^2+\lambda}v \). Therefore, Policy 3-3 applies to this interval.

Indeed, Policy 1 is optimal for \( \frac{1-\lambda(1-\sqrt{4k^2+\lambda})}{k^2+\lambda}v < p \leq v \).

We summarize the results of the comparisons below:

- If \( \lambda < k < 1 \) and \( p \in (\frac{1+\lambda}{k+1}v, \frac{2k^4+2k^3-2k^2\lambda-\lambda^2+\lambda\lambda\sqrt{4k^3+4k^2+\lambda}}{2k^3(k+1)}v) \), then Policy 2 is optimal.
- If \( \lambda < k < 1 \) and \( p \in (\frac{2k^4+2k^3-2k^2\lambda-\lambda^2+\lambda\lambda\sqrt{4k^3+4k^2+\lambda}}{2k^3(k+1)}v, \frac{k^2(k+1)+\lambda(\lambda+1)}{(k+1)(k^2+\lambda)}v) \), then Policy 3-1 is optimal.
- If \( \lambda < k, 1 < k < \lambda + 1 \), and \( p \in (\frac{1+\lambda}{k+1}v, \frac{k^2(k+1)+\lambda(\lambda+1)}{(k+1)(k^2+\lambda)}v) \), then Policy 2 is optimal.
- If \( \lambda < k, 1 < k < \lambda + 1 \), and \( p \in (\frac{k^2(k+1)+\lambda(\lambda+1)}{(k+1)(k^2+\lambda)}v, \frac{2k^4+2k^3-2k^2\lambda-\lambda^2+\lambda\lambda\sqrt{4k^3+4k^2+\lambda}}{2k^3(k+1)}v) \), then Policy 3-2 is optimal.
- If \( k > \lambda + 1 \) and \( p \in (\frac{1+\lambda}{k+1}v, \frac{k^2(k+1)+\lambda(\lambda+1)}{(k+1)(k^2+\lambda)}v) \), then Policy 2 is optimal.
- If \( k > \lambda + 1 \) and \( p \in (\frac{k^2(k+1)+\lambda(\lambda+1)}{(k+1)(k^2+\lambda)}v, \frac{1-\lambda(1-\sqrt{4k^2+\lambda})}{k^2+\lambda}v) \), then Policy 3 is optimal.
- If \( k > \lambda + 1 \) and \( p \in (\frac{1-\lambda(1-\sqrt{4k^2+\lambda})}{k^2+\lambda}v, \frac{1-\lambda(1-\sqrt{4k^2+\lambda})}{k^2+\lambda}v) \), then Policy 1 is optimal.

**Proof of Corollary 2:** See the proof of Proposition 5.