Reducing Waste of Perishables in Retailing through Transshipment

Qing Li Peiwen Yu

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Abstract

Transshipment in retailing is a practice where one outlet ships its excess inventory to another outlet with inventory shortages. By balancing inventories, transshipment can reduce waste and increase fill rate at the same time. In this paper, we explore the idea of transshipping perishable goods in offline retailing. In the offline retailing of perishable goods, customers typically choose the newest items first, which can lead to substantial waste. We show that in this context, transshipment plays two roles. One is inventory balancing, which is well known in the literature. The other is inventory separation, which is new to the literature. That is, transshipment allows a retailer to put newer inventory in one outlet and older inventory in the other. This makes it easier to sell older inventory and reduces waste as a result. Our numerical studies show that transshipment and clearance sales are substitutes in terms of both increasing profit and reducing waste. In particular, transshipment can increase profit by up to several percentage points. It is most beneficial in increasing profit when the variable cost of products is high and hence few items are put on clearance sale. Although transshipment does not always reduce waste, when it does, the reduction can be substantial. Similar to the way it impacts profit, transshipment can reduce waste the most when the variable cost of products is high and hence products are too expensive to be put on clearance sale.

1 Introduction

The sale of perishable products accounts for over 50% of the business of grocery retailing in many countries. As consumers become increasingly health conscious, the importance of perishable products can only grow. Besides, retailers also rely on perishable products to drive store traffic and gain competitive edge (Tsiros and Heilman 2005). Managing inventory of perishable products, however, is challenging and waste is substantial in retailing. Retailers remove items from the shelves when they are near or past their expiration dates. In America, approximately 40% of food produce is wasted, much of it in retailing (Kaye 2011). Waste in
retailing is a problem in many other places too. According to a recent study by Friends of the Earth, the four main supermarkets in Hong Kong throw away 87 tons of food a day, and most of the waste ends up in landfill (Wei 2012). Many retailers have programs aimed at addressing the problem. For example, the Food Waste Reduction Alliance in the United States, the Waste and Resources Action Programme in the United Kingdom, and the Retailers’ Environmental Action Programme in Europe were all established with waste reduction as their primary goal.

Waste at the retailer level, however, may or may not be the biggest part of the problem. In the process of moving food from farm to fork, the waste upstream and at the consumer level may sometimes be much more significant. However, as value is added along the supply chain, the economic losses from the waste upstream, while substantial, are less pronounced. Furthermore, retail industries are highly consolidated and typically dominated by a few retail giants. Modest waste reduction in one supermarket chain can be substantial. Retailers wield strong influence over their suppliers and customers. Waste reduction initiatives started by a retailer can have spillover effects on their suppliers and customers. Not only can they reduce waste generated at the retailer level, they can raise awareness among suppliers and customers and motivate them to reduce waste as well (Kor et al. 2017). Over the years, much effort has been put into waste reduction in retailing. Nevertheless a lot more can still be done.

The main challenges in waste reduction at the retailer level are well known. First, demand is uncertain, and hence it is difficult to match supply with demand. Second, perishable items typically have very short lifetimes and hence they need to be sold in a short time window. Third, customers choose items on a last-in-first-out (LIFO) basis. Retailers replenish their shelves periodically. Whenever a new shipment arrives, the items on the shelves, which have shorter remaining lifetimes, are one step closer to the bin. Various ideas for waste reduction have been adopted in practice and discussed in the academic literature, including the use of technology to extend lifetime, clearance sales, and frequent stock rotation. In this paper, we add a new weapon to the arsenal in the war against waste: transshipment.

Transshipment has been widely studied in the operations management literature, but it has not been studied in retailing of perishable products under the LIFO rule. Existing research shows that its benefit comes from balancing inventories across different locations and hence reducing waste at some locations and shortages at others at the same time. In this study we explore the idea of transshipment in an offline retailer consisting of two outlets. The retailer replenishes its perishable products every period and at the end of each period, the retailer can either put the products that have not expired on clearance sale or carry them over to the
next period. The retailer can also transship them from one outlet to the other. Our analysis shows that in this context, transshipment works very differently. It can balance inventories across different locations, similar to what is known in the literature. However, besides that, transshipment also plays a very different role in that it allows the separation of items with different remaining lifetimes. That is, transshipment allows the retailer to put newer inventory in one outlet and older inventory in the other. This makes it easier to sell older inventory and reduces waste as a result. For an approximation that relies on only two pieces of information, namely the number of items expiring in one period (old items) and that of the rest (new items), we show that the optimal policy can be characterized by two increasing switching curves. The two switching curves divide the entire state space into three regions. In the first region, only one outlet holds old inventory while both hold new inventory. In the second, one outlet holds old inventory and the other holds new inventory. In the third, one outlet holds new inventory while both hold old inventory.

Our numerical studies show that transshipment and clearance sales are substitutes in terms of both increasing profit and reducing waste. In particular, transshipment can increase profit by as much as several percentage points when the variable cost of products is high and hence few items are put on clearance sale. Although transshipment does not always reduce waste, when it does, the reduction can be substantial. Similar to its impact on profit, transshipment can reduce waste the most when the variable cost of products is high and hence products are too expensive to be put on clearance sale.

To turn the idea of transshipment of perishables into reality, three important issues must be considered. First, since transshipment may not always increase profit significantly, waste reduction has to offer an additional incentive for retailers. The incentive ultimately comes from environmentally conscious consumers who may choose to shop at greener retailers. Environmental agencies and organizations may periodically conduct waste auditing and publish their findings so that consumers can make informed choices. Second, the value of perishables per unit is typically low. Therefore, transshipment is viable economically only if the scale is large enough and the logistics is extremely efficient. Ideally, the transshipment should be integrated with the existing replenishment process so that the additional variable cost is minimal. Third, transshipment between outlets may require the cooperation of store managers whose incentive may not be aligned with that of the retailer. All issues can be difficult, but not impossible, barriers to overcome. How to overcome these barriers is beyond the scope of our current research. Instead, we focus on how transshipment should be implemented and its impact on profit and
waste. We believe this is the first step and only then will we know whether it is worth over-
coming these barriers in implementation. Practically, our results are directly useful for retailers
of perishable goods, especially those that consider sustainability a priority. Theoretically, the
study also provides a completely new perspective on transshipment, an important concept in
the field of operations management, and enriches the literature on perishable inventory and
that on transshipment.

The remainder of the paper is organized as follows. In Section 2, we review related literature.
We present the general formulation in Section 3. The general problem is computationally
challenging. Therefore, we discuss its approximation in 4. The effects of transshipment on
profit and waste are tested numerically in Section 5. We discuss an extension in Section 6 and
conclude the paper in Section 7.

2 Related Literature

The study is directly related to two streams of literature in operations management. The first
is the literature on transshipment. This literature is voluminous. The recent studies can be
classified into two types. In the first type, there is a central decision maker who has access to full
information and makes all the decisions. Representative studies of this type include, Abouee-
Mehrizi et al. (2015) (lost sales), Hu et al. (2008) and Li and Yu (2014) (capacity constraints),
and Yang and Zhao (2007) (virtual transshipment). In all these studies, the objective is to
characterize and compute the optimal replenishment and transshipment policies. In the second
type of studies, there are multiple decision makers with different incentives. Various research
questions have been raised. For example, Hu et al. (2007) focused on the question of whether
a pair of coordinating transshipment prices, i.e., payments that each party has to make to the
other for the transshipped goods, can be set globally such that the local decision makers are
induced to make inventory and transshipment decisions that are globally optimal. Dong and
Rudi (2004) and subsequently Zhang (2005) studied how transshipments affect independent
manufacturers and retailers in a supply chain where retailers can transship inventory. Studies
also exist that consider the cooperation and competition of retailers using cooperative game
theory (e.g., Sosic 2006, Fang and Cho 2014). None of these studies has considered perishable
products with a general lifetime.

The second stream is the literature on perishable inventory. A considerable renewed interest
exists in the area (see, for example, Chao et al. 2018, Chao et al. 2015, Chen et al. 2014, Li
and Yu 2014, and reviews by Karaesmen et al. 2011 and Nahmias 2011). Particularly related
to our study is the strand of literature that considers the LIFO rule. Cohen and Pekelman (1978) analyzed the evolution over time of the age distribution of inventory. Under two particular order policies, constant order quantity and fixed critical number, they determined the shortages and outdates in each period by the age distribution and related them to inventory decisions. Pierskalla and Roach (1972) and Deniz et al. (2010) considered issuing endogenously and the set of feasible issuing rules includes LIFO. The former showed that under most of the objectives, first-in-first-out (FIFO) is the optimal issuing rule. The latter focused on finding heuristics to coordinate replenishment and issuing. Parlar et al. (2008) and Cohen and Pekelman (1979) compared FIFO issuance with LIFO issuance. None of the above-mentioned papers has considered the optimal inventory ordering policy under LIFO.

In spite of the practical relevance of the LIFO rule to retailing, little work has been done, especially in terms of optimal policies, perhaps due to the technical difficulties. However, recent progress is encouraging. Li et al. (2016) focused on the optimal policies on inventory control and clearance sales under LIFO and a general life time. They showed that a clearance sale may occur if the level of inventory with a remaining lifetime of one period is either very high or very low, a phenomenon that is unique to the LIFO rule. Furthermore, they showed that myopic heuristics requiring only information about total inventory and information about the inventory with a remaining lifetime of one period performed consistently well. Li et al. (2017) examined the impact of shelf-life-extending packaging on the optimal policy, cost, and waste. One interesting insight they gave was that although it may not be optimal in terms of cost, the adoption of shelf-life-extending packaging can consistently reduce waste substantially. None has considered transshipment in the literature on perishable inventory with a general lifetime.

The study closest to ours is perhaps that of Zhang et al. (2017). They studied transshipment of perishable inventory with a general lifetime between two locations. However, they assumed a FIFO rule and exogenous order-up-to levels, neither of which holds in retailing. In summary, we are the first to consider transshipment in perishable inventory management in retailing.

3 The General Formulation

There are two identical outlets, indexed by superscript $i = 1, 2$, owned by the same retailer. The products they sell have an $n$-period lifetime. The products can be sold either at a regular price, $p$, or a clearance sale price, $s$. Under a regular price, the demand at each outlet is random and is modeled by random variable $D^i$. The demand under a clearance sale is so high (or $s$ is so low) that inventory on clearance sales will never go unsold. More sophisticated pricing schemes
have been used in services such as hotels and airlines, but are uncommon in offline retailing. We assume that $D^1$ and $D^2$ are identically but not necessarily independently distributed. The assumption is made so that we can sharpen the key insights and we will discuss the more general cases toward the end.

The timing of events is as follows. 1) At the beginning of a period, the retailer determines how much to order and how much and what should be transshipped from one outlet to the other. 2) Then the random demand for regular sales is realized. 3) At the end of the period, the unsold inventory with a remaining lifetime of one period expires; and 4) the retailer determines how much of the inventory that has not expired should be carried over to the next period and how much should be put on clearance sale. Because there is no information updating between the ordering and transshipment decisions in 1) and the clearance sale decisions in 4), we redefine a period by moving 4) to the beginning of a period. In other words, all decisions are made at the beginning of a period. We assume that there is no transshipment cost in the model and the implication of transshipment cost will be discussed in the Conclusion section.

For outlet $i$, the initial inventory is represented by a vector $x_i^i = (x_{i1}, x_{i2}, ..., x_{in-1})$, where $x_{ij}$ represents the inventory with a remaining lifetime of $j$ periods at outlet $i$. Let $x_j = x_{j1} + x_{j2}$. The system state can be captured by $x = (x_1, x_2, ..., x_{n-1})$. Let $q^i$ be the order quantity of new items at outlet $i$. Let $z^i = (z_{i1}, ..., z_{n-1})$, where $z_{ij}$ is the inventory with a remaining lifetime of $j$ periods that retail outlet $i$ has after transshipment and clearance sale. As such, the total amount of inventory with a remaining lifetime of $j$ periods available for regular sales is $z_{j1}^i + z_{j2}^i$ and the amount sold in clearance sales is $x_j - z_{j1}^i - z_{j2}^i$. Customers would always choose the freshest products first; that is, inventory leaves the retail shelf on a LIFO basis. Suppose the system state becomes $Y^i(q^i, z^i, D^i) = (Y_1^i, Y_2^i, ..., Y_{n-1}^i)$ in the next period. Then, for $1 \leq j \leq n-2$

$$Y_j^i(q^i, z^i, D^i) = (z_{j+1}^i - (D^i - q^i - \sum_{k=j+2}^{n-1} z_{ik})^+)$$

and

$$Y_{n-1}^i(q^i, z^i, D^i) = (q^i - D^i)^+.$$ 

The outdated amount is

$$S(q^i, z^i, D^i) = (z_1^i - (D^i - q^i - \sum_{j=2}^{n-1} z_{ij})^+)^+.$$
Let \( c, \theta, \) and \( \alpha \) be the ordering cost, outdating cost, and the discounting factor, respectively. Without loss of generality, we assume that there is no holding cost. The dynamic programming formulation is as follows:

\[
J_t(z^i, q^i) = -s \sum_{j=1}^{n-1} z^i_j - c q^i + p \mathbb{E} \min(q^i + \sum_{j=1}^{n-1} z^i_j, D^i) - \theta \mathbb{E} S(q^i, z^i, D^i),
\]

(1)

and

\[
v_t(x) = s \sum_{j=1}^{n-1} x_j + \max\{J_t(z^1, q^1) + J_t(z^2, q^2) + \alpha \mathbb{E} v_{t+1} \sum_{i=1}^{2} Y^i(q^i, z^i, D^i)\},
\]

(2)

subject to \( z^1_j + z^2_j \leq x_j, \quad z^i_j \geq 0, \quad q^i \geq 0 \) for all \( i = 1, 2 \) and \( j = 1, 2, ..., n - 1 \). On the right-hand side of (1), the second term is the purchasing cost, the third term the revenue from regular sales, and the last term the outdating cost. The sum of the first terms on the right-hand sides of (1) and (2) represents the revenue from clearance sales. Hence \( J_t(z^i, q^i) \) is the one-period profit generated at outlet \( i \). Denote by \( (\bar{z}^i_1, \bar{q}^i_1), (\bar{z}^i_2, \bar{q}^i_2), ..., (\bar{z}^i_{n-1}, \bar{q}^i_{n-1}) \) the optimal solution to (2).

Let \( e_i \) denote an \( n - 1 \) dimensional unit vector where the \( i \)-th element equals one and all other elements equal zero. Let \( \delta \) be a small positive number. We can show the following results on the marginal values of initial inventories.

**Lemma 1**

(i) \( v_t(x + \delta e_i) \leq v_t(x + \delta e_{i+1}) \);

(ii) \( s\delta \leq v_t(x + \delta e_i) - v_t(x) \leq c\delta \);

(iii) \( J_t(z^i, q^i) \) is submodular in \( (z^i_1, q^i) \) and \( (z^i_1, z^i_j) \) for \( j \geq 2 \).

In the next theorem, we show that if items with a two-period or longer lifetime are sold through clearance sales under the optimal policy, then all the older inventories are cleared and no new items are ordered.

**Theorem 1** If \( \bar{z}^1_i + \bar{z}^2_i < x_i \) for some \( i \geq 2 \), then

(i) \( \bar{z}^1_j = \bar{z}^2_j = 0 \) for all \( j < i \);

(ii) \( \bar{q}^1 = \bar{q}^2 = 0 \).
We show in the following theorem that if the total inventory on hand with a remaining lifetime of at least two periods is large enough, then at least one of the two outlets will not be keeping inventory with a remaining lifetime of one period. Let \( l_0 = \Phi^{-1}\left(\frac{p-s}{\theta}\right) \), which represents the optimal quantity of inventory with a remaining lifetime of one period an outlet should carry over to the next period when the selling price is \( p \), disposal cost is \( \theta \), and the opportunity cost (clearance price) is \( s \).

**Theorem 2** If \( \sum_{i=2}^{n-1} x_i \geq 2l_0 \), then either \( \bar{z}_1^1 = 0 \) or \( \bar{z}_1^2 = 0 \).

If the optimal solution is symmetrical, then both \( \bar{z}_1^1 \) and \( \bar{z}_1^2 \) are zero. However, the optimal solution may not be symmetrical, even though the two outlets are identical and face identically distributed demands.

## 4 Approximations

The structural properties in Section 3 provide useful guidance. However, to put the ideas into practice, there are still open questions. First, how much should each outlet order in each period, and how much existing inventories should be sold in clearance sales and how much should be carried over to the next period? Second, how should the existing inventories be allocated between the two outlets? Third, what would be the impact of transshipment on profit and waste? To answer these questions with the general formulation in Section 3, we need to know how many units of inventory there are in each age group, and with that information, to solve a dynamic program with a multi-dimensional state space and a non-concave objective function. The former is impossible given the current bar code design and standard and the latter is challenging computationally. Approximation is the only way forward.

Based on the ideas from Li et al. (2016), we simplify the general formulation in two steps. First, we approximate the profit-to-go by a linear function. That is, we let \( v_t(x) = v \sum_{j=1}^{n-1} x_j \), where \( v \) is a number bounded by \( c \) and \( s \) (e.g., \( v = (s + c)/2 \)) because the marginal value of inventory is bounded by \( c \) and \( s \). Second, we aggregate the state variables \( (x_2, x_3, ..., x_{n-1}) \); that is, we look for policies that rely only on \( x_1 \) and the sum of \( x_2, x_3, ..., x_{n-1} \).

In this section, we continue to use \( x_1 \) to represent the inventory with a remaining lifetime of one period, but use \( x_2 \) to represent the total inventory with a remaining life time of two periods or longer. For ease of exposition, we call the former old inventory and the latter new inventory. We use \( z_1^1 \) and \( z_2^2 \) to represent the amount of old inventory and new inventory, respectively, at outlet \( i \) on regular sale. Let \( y^i \) be the amount of new inventory after ordering at outlet \( i \).
is, $y^i$ is the order-up-to level for new inventory at outlet $i$. To avoid the need for additional notation, we continue to use $J_t$ and $v_t$ to represent respectively the one-period profit for an outlet and the total maximal profit when the above approximations are used. Essentially, we are solving the following optimization program:

$$J_t(z_i^1, y^i) = -sz_i^1 - cy^i + pE\min(D^i, z_i^1 + y^i) - \theta E(z_i^1 - (D^i - y^i)^+) + \alpha v(y^i - D^i)^+,$$

and

$$v_t(x_1, x_2) = s(x_1 + x_2) + \max\{(c - s)(z_2^1 + z_2^2) + J_t(z_2^1, y^1) + J_t(z_2^2, y^2)\}$$

subject to $z_j^1 + z_j^2 \leq x_j$, $z_j^i \geq 0$, $y^i \geq z_i^2$ for all $i = 1, 2$ and $j = 1, 2$. Denote by $(\bar{z}_i^1, \bar{z}_i^2, \bar{y}^i)$ the optimal solution to (3).

It is important to point out that the properties related to the general problem in Section 3 continue to hold true after the approximations. Specifically, the marginal value of inventory is bounded by $c$ and $s$, new inventory is more valuable than old inventory (Lemma 1), old inventory should be cleared before new inventory, and ordering occurs only when new inventory is not sold in clearance sales (Theorem 1).

The first part of the following lemma is similar to Lemma 1 (iii), but the approximations yield a stronger result.

**Lemma 2**

(i) $J_t(z_i^1, y^i)$ is submodular in $(z_i^1, y^i)$;

(ii) If $D^i$ has a PF$_2$ distribution, then $J_t(z_i^1, y^i)$ is quasiconcave in $y^i$.

PF$_2$ distributions are also known to have log-concave densities (Ross 1983). This is a common assumption in the inventory literature (e.g., Huggins and Olsen 2010, Li et al. 2016), and the class of distributions includes many commonly used distributions. PF$_2$ distributions have the following smoothing property: if $D$ is a PF$_2$ random variable and $f(x)$ is quasiconcave, then $E f(x - D)$ is quasiconcave.

Since the two outlets are identical and are facing identically distributed demands, one might expect the optimal solution to be symmetrical. Indeed, when the inventory is depleted on an FIFO basis, we can show that there is a symmetrical optimal solution. However, this is not the case in our setting.
Theorem 3  Under the optimal policy, at least one of the two outlets holds either the old inventory or the new inventory, but not both types of inventory; that is, at least one of \( \bar{z}^1_1, \bar{z}^2_1, \bar{y}^1 \) and \( \bar{y}^2 \) is zero.

Theorem 4 shows that the presence of old inventory leads to asymmetrical optimal solution.

Theorem 4  Suppose that \( x_1 > 0 \) and \( x_2 > 0 \).

(i) (Separation of Inventories) When \( \bar{z}^1_1 + \bar{z}^2_1 > 0 \), there must be an asymmetric optimal solution; that is, \((\bar{z}^1_1, \bar{y}^1) \neq (\bar{z}^2_1, \bar{y}^2)\).

(ii) (Balance of Inventories) When \( \bar{z}^1_1 + \bar{z}^2_1 = 0 \), there must be a symmetric optimal solution; that is, \((\bar{z}^1_1, \bar{y}^1) = (\bar{z}^2_1, \bar{y}^2)\).

Theorem 4 shows that whether the retailer uses transshipment to separate inventories or balance inventories depends on whether there is any inventory with a remaining lifetime of one period in the system. In what follows, we identify conditions under which all of that inventory is cleared.

Theorem 5

(i) If \( x_2 \geq 2u_0 \), then it is optimal to clear all of the old inventory, clear the new inventory down to \( 2u_0 \) and allocate that equally between the two outlets, and order nothing; that is, \( \bar{y}^1 = \bar{y}^2 = \bar{z}^1_2 = \bar{z}^2_2 = u_0 \), and \( \bar{z}^1_1 = \bar{z}^2_1 = 0 \).

(ii) There exists an increasing function \( A(x_2) \) such that if \( x_1 \leq A(x_2) \), then all \( x_1 \) is cleared.

Part (i) of Theorem 5 is related to Theorem 2. But the approximation allows for a stronger result. In light of Theorem 4, if \( x_1 \) is not too small and \( x_2 \) is not too large, then there is an asymmetric optimal solution (i.e., separation of inventories). Otherwise, the two outlets hold exactly the same level of inventory after transshipment (i.e., balance of inventories). Because of Theorems 1 and 3, the optimization problem (3) is equivalent to the following:

\[
v_t(x_1, x_2) - s(x_1 + x_2) = \max \{ K_1(x_1, x_2), K_2(x_1, x_2), K_3(x_1, x_2) \} \tag{4}
\]

where

\[
K_1(x_1, x_2) = \max_{z^1_1 + z^2_1 \leq x_1, y^1 \geq x_2, z^1_1 \geq 0} \{ (c - s)x_2 + J_t(z^1_1, y^1) + J_t(z^2_1, 0) \},
\]

\[
K_2(x_1, x_2) = \max_{0 \leq z^2_1 \leq x_1, y^1 + y^2 \geq x_2, y^2 \geq 0} \{ (c - s)x_2 + J_t(0, y^1) + J_t(z^2_1, y^2) \},
\]
and
\[ K_3(x_1, x_2) = \max_{z_1^1 + z_2^2 \leq x_2, z_1^1 \geq 0} \{(c - s)(z_1^1 + z_2^2) + J_t(0, z_1^1) + J_t(0, z_2^2)\} \]

Here \( K_1 \) represents the case when there are no clearance sales of new items, and all new items are allocated to outlet 1. \( K_2 \) represents the case when there are no clearance sales of new items, and all old items are allocated to outlet 2. \( K_3 \) presents the case when some or all new items are sold in clearance sales, in which case, no order is placed and all old items are sold in clearance sales. In light of Theorems 1 and 3, these events are collectively exhaustive. Let \( u = \arg \max_{y \geq 0} J_t(0, y) \). Then, \( u = \Phi^{-1}(\frac{p-c}{\rho - \alpha v}) \), which represents the optimal ordering quantity of new items when the initial inventory of old items is zero. Let \( u_0 = \arg \max_{y \geq 0} \{(c - s)y + J_t(0, y)\} \). Then \( u_0 = \sup \{x : \Phi(x) \leq \frac{p-c}{\rho - \alpha v}\} \).

The optimal separation policy is given in the next theorem.

**Theorem 6** There exist two increasing functions \( B(x_1) \geq C(x_1) \) such that

(i) If \( x_2 > B(x_1) \), then at most one outlet holds old inventory while both outlets hold new inventory;

(ii) If \( x_2 < C(x_1) \), then at most one outlet holds new inventory while both outlets hold old inventory;

(iii) If \( C(x_1) \leq x_2 \leq B(x_1) \), then one outlet holds old inventory while the other holds new inventory.

Theorem 6 provides an optimal mapping between the allocation of inventories and the state. The two outlets can take turn to be the one which receives the old inventory so that in the long run, the two outlets have equally fresh inventories. We further characterize the structure of the optimal clearance sales policy of old items in the following theorem.

**Theorem 7**

(i) For \( x_2 < C(x_1) \), the old inventory is put on clearance sales if and only if \( x_2 \geq 2l_0 - x_1 \).

(ii) For \( C(x_1) \leq x_2 < B(x_1) \), the old inventory is put on clearance sales if and only if \( x_1 \geq l_0 \).

(iii) For \( x_2 > B(x_1) \), there exists a function \( e_0(x_1) \) such that the old inventory is put on clearance sales if and only if \( x_2 \geq e_0(x_1) \).
Figure 1: Separation of Inventories and Clearance Sales (Note: Clearance and Non-clearance Regions are Labeled by “C” and “NC”)

The results in Theorems 6 and 7 are shown in Figure 1. The clearance and non-clearance regions are labeled by “C” and “NC”, respectively. In Figure 1, in the region between $C(x_1)$ and $B(x_1)$, one outlet holds old inventory and the other holds new inventory; that is, old and new inventories are separated. In this case, if $x_1$ is greater than $l_0$, then $x_1$ is cleared down to $l_0$; otherwise, there is no clearance sale of $x_1$. This is consistent with Theorem 4 where we show that inventories are separated if and only if there is old inventory in the system after clearance sales. In general, old items are more likely to be put on clearance sale as $x_2$ increases. We can also see from the figure that for a given $x_2$, when $x_1$ is small enough, all $x_1$ should be cleared.

The structure of the optimal ordering policy is given in the following theorem.

**Theorem 8**

(i) If $x_2 < C(x_1)$, there exists a function $e_1(x_2) \leq 2l_0 - x_1$ such that new items are ordered if and only if $x_1 \leq e_1(x_2)$.

(ii) If $C(x_1) \leq x_2 < B(x_1)$, new items are ordered if and only if $x_2 \leq u$.

(iii) If $x_2 > B(x_1)$, there exists a decreasing function $e_2(x_1)$ such that new items are ordered if and only if $x_2 \leq e_2(x_1)$. 

The results in Theorem 8 can be visualized in Figure 2. The ordering and non-ordering regions are labeled by “O” and “NO”, respectively. The optimal ordering quantity is monotonically decreasing if the inventory level of old items increases, but not necessarily decreasing when there are more new items. Similar results have been established by Li et al. (2016). Technically, this happens because even though $J_t(z^i_1, y^i)$ is quasiconcave in $y^i$, $\max_{z^i_1 \geq 0} J_t(z^i_1, y^i)$ is not necessarily quasiconcave in $y^i$.

5 Numerical Studies

In this section, we conduct numerical studies to investigate two related issues. First, clearance sales and transshipment are two tools for retailers to fight perishability, but how are they related in creating value in terms of increasing profit and reducing waste? Second, under what circumstances does transshipment create the most value in terms of increasing profit and reducing waste? For all the numerical studies, when the lifetime is 3, we compute the optimal policy for the dynamic programming formulation in Section 3, but for any longer lifetimes, we compute the optimal policy for the approximated formulation in Section 4.
5.1 Value of transshipment and clearance sales

In this subsection, we study the value of transshipment and clearance sales in terms of both improving profit and reducing waste. We let the lifetime be 3 and calculate the profit and waste under the optimal policy of the dynamic program at initial state (0, 0) for four different scenarios depending on whether or not transshipment and/or clearance sales are adopted. The results are presented in Table 1.

Table 1: Value of transshipment and clearance sales.

<table>
<thead>
<tr>
<th>c, θ, s</th>
<th>Transshipment</th>
<th>Profit</th>
<th>Clearance</th>
<th>Y</th>
<th>N</th>
<th>Diff</th>
</tr>
</thead>
<tbody>
<tr>
<td>Y</td>
<td>174.76</td>
<td>174.16</td>
<td>0.60</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>N</td>
<td>174.08</td>
<td>173.16</td>
<td>0.92</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Diff</td>
<td>0.68</td>
<td>1.00</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Waste</td>
<td>Y</td>
<td>2.85</td>
<td>2.49</td>
<td>0.36</td>
<td></td>
<td></td>
</tr>
<tr>
<td>N</td>
<td>4.12</td>
<td>4.37</td>
<td>-0.25</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Diff</td>
<td>-1.27</td>
<td>-1.88</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Note. T = 15, r = 10, α = 0.99, D = Uniform[0, 2].

There are two observations from the table. First, clearance sales and transshipment seem to be substitutes in both improving profit and reducing waste; i.e. having clearance sales decreases the value of having transshipment in improving profit or reducing waste. Second, in the presence of clearance sales, transshipment may not always reduce waste. This happens for example when \( c = 3, \theta = 1, \) and \( s = 1. \) Since transshipment may reduce the need for clearance sales,
sales, the outlets may carry more old inventory in the future periods with transshipment. In addition, the retailer may order more when transshipment is used. Both may increase waste.

5.2 The impact of transshipment on profit and waste

For \( n = 3 \), we can calculate the differences in profit and waste between the optimal policy of the dynamic program with transshipment and that without. For \( n \geq 4 \), which might be more realistic, computing the optimal policies of the dynamic program is no longer possible. Instead, we compare the performance of the heuristic policy with transshipment against that of the same heuristic policy but without transshipment (i.e., the two outlets are managed independently using the same myopic policies). We evaluate the performance of approximations in terms of both profit and waste. We consider a 15-period horizon, i.e. \( T = 15 \). The values taken by system parameters are: \( n \in \{4, 5\} \), \( c \in \{3, 5, 7\} \), \( s \in \{0, 1\} \) and \( \theta \in \{0, 1, 2\} \). The discount rate \( \alpha = 0.99 \), and the regular sales price \( r \) is held constant at 10.

When \( n \geq 4 \), an important issue when using the approximation is that the approximation only tells us the amount of total new inventory \( \bar{z}_i^2 \) at each location \( i \). However, there are many ways to allocate inventory across the two outlets such that \( \sum_{j=2}^{n-1} z_i^j = \bar{z}_i^2 \) for \( i \in \{1, 2\} \) and \( \sum_{i=1}^2 z_i^j \leq x_j \). The way inventory is allocated affects the amount of the oldest inventory in the next period and in turn the profit and waste.

We use the sequential method in Algorithm 1 to allocate inventory to maximize “separability”. The idea is to allocate the freshest inventory \( x_{n-1} \) to one outlet as much as possible, and then allocate the second freshest inventory \( x_{n-2} \) to the other outlet as much as possible, and the process continues for \( x_{n-3}, \ldots, x_2 \) until all new inventory \( \sum_{i=1}^2 \bar{z}_i^2 \) are exhausted.

We use three examples to illustrate the algorithm for \( n = 4 \). In all examples, we assume that the new inventory in two outlets under the approximate policy is \((\bar{z}_1^2, \bar{z}_2^2) = (9, 4)\). In our first example, suppose \((x_2, x_3) = (12, 10)\). The algorithm starts with the freshest inventory \( x_3 \), since \( \bar{z}_2^2 < \bar{z}_1^2 < x_3 \), outlet 1 will get 9 units and outlet 2 will get the remaining 1 unit. After this step, the remaining inventories to be allocated to outlets 1 and 2 are 0 and 3 units, respectively. The final allocation given by the algorithm is \((z_1^1, z_1^3) = (9, 0)\) and \((z_2^2, z_2^3) = (3, 1)\).

In the second example, suppose \((x_2, x_3) = (12, 6)\). Since \( \bar{z}_2^2 < x_3 < \bar{z}_1^2 \), Algorithm 1 first allocates \( z_1^3 = 6 \) and \( z_2^3 = 0 \). The remaining inventories to be allocated to outlets 1 and 2 are 3 units and 4 units, respectively. The final allocation is \((z_1^1, z_1^3) = (3, 6)\) and \((z_2^2, z_2^3) = (4, 0)\). In the third example, suppose \((x_2, x_3) = (12, 3)\). Since \( x_3 < \bar{z}_2^2 < \bar{z}_1^1 \), the resulting allocation from Algorithm 1 is \((z_1^1, z_1^3) = (9, 0)\) and \((z_2^2, z_2^3) = (1, 3)\).
Algorithm 1 Allocation of new inventory across the two outlets

Initialize $j = n - 1$

while $j \geq 2$

- If there exists $i$ such that $\bar{z}_2^{3-i} \leq z_i \leq x_j$, then let $z_i^j = \bar{z}_2^i$ and $z_j^{3-i} = \min\{x_j - \bar{z}_2^i, \bar{z}_2^{3-i}\}$
- If there exists $i$ such that $\bar{z}_2^{3-i} \leq x_j \leq \bar{z}_2^i$, then let $z_i^j = x_j$ and $z_j^{3-i} = 0$
- If there exists $i$ such that $x_j \leq \bar{z}_2^i \leq \bar{z}_2^{3-i}$, then let $z_i^j = x_j$ and $z_j^{3-i} = 0$
- $\bar{z}_2^i \rightarrow \bar{z}_2^i - z_i^j$ for $i = 1, 2$
- $j \rightarrow j - 1$

end while

We summarize the numerical results in Table 2. In these studies, the increase in profit as a result of transshipment can be more than 5%, and transshipment can increase profit the most when the variable cost is high (or profit margin is low). The increase is nontrivial because retailing of perishables is a low-margin business. Although in many cases transshipment increases profit only slightly (by less than 1%), it can significantly reduce waste in most cases (by more than 5%). Similar to its impact on profit, transshipment reduces waste the most (by more than 20%) when the variable cost is high because that is when the opportunity cost of clearance sale is high. As we mentioned earlier, however, transshipment may not always reduce waste. In one case, the increase in waste is 58% (when $c = 3, n = 4, \theta = 1, s = 1$). In this case, the waste under no transshipment is only 0.37 and a slight increase in absolute waste as a result of transshipment leads to a very large increase in percentage. Overall, increases in waste as a result of transshipment are rare.

6 Non-identically Distributed Demands

In the earlier analysis, to avoid the obfuscation of main insights, we have assumed that the two outlets face identically distributed demands. Now let us consider a more realistic case where one outlet faces a stochastically larger demand than the other. Without loss of generality, suppose $D^2$ is stochastically larger than $D^1$ (Ross 1983); that is, $D^2 \succeq_{st} D^1$. According to our analysis, it is not a good idea to put old and new inventory in the same outlet. When the demands are not identically distributed, a new issue arises: which outlet should get the new inventory and which outlet should have the old inventory? On the one hand, it makes sense to allocate the old inventory to outlet 2 because its stochastically larger demand makes it more likely to deplete the old inventory. On the other hand, outlet 2 requires a higher total inventory to fill its larger
Table 2: Percentage Differences in Profit and Waste between the Heuristic with Transshipment and That without.

<table>
<thead>
<tr>
<th>n = 3</th>
<th>( c = 3 )</th>
<th>( c = 5 )</th>
<th>( c = 7 )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( s = 0 )</td>
<td>( s = 1 )</td>
<td>( s = 0 )</td>
</tr>
<tr>
<td>Profit</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \theta = 0 )</td>
<td>0.47</td>
<td>0.33</td>
<td>1.09</td>
</tr>
<tr>
<td>( \theta = 1 )</td>
<td>0.48</td>
<td>0.34</td>
<td>1.04</td>
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<td>( \theta = 2 )</td>
<td>0.68</td>
<td>0.27</td>
<td>1.20</td>
</tr>
<tr>
<td>Waste</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \theta = 0 )</td>
<td>-5.46</td>
<td>-5.05</td>
<td>1.87</td>
</tr>
<tr>
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<tr>
<td>( \theta = 2 )</td>
<td>-5.17</td>
<td>-10.00</td>
<td>-13.63</td>
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</tbody>
</table>

<table>
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<th>( c = 7 )</th>
</tr>
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<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Profit</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \theta = 0 )</td>
<td>0.27</td>
<td>0.29</td>
<td>0.42</td>
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<td>( \theta = 1 )</td>
<td>0.37</td>
<td>0.32</td>
<td>0.70</td>
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<tr>
<td>( \theta = 2 )</td>
<td>0.53</td>
<td>0.24</td>
<td>0.94</td>
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<tr>
<td>Waste</td>
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<td></td>
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<tr>
<td>( \theta = 0 )</td>
<td>-8.03</td>
<td>-8.12</td>
<td>-14.95</td>
</tr>
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<td>( \theta = 1 )</td>
<td>-7.75</td>
<td>58.48</td>
<td>-14.81</td>
</tr>
<tr>
<td>( \theta = 2 )</td>
<td>-8.87</td>
<td>NaN</td>
<td>-20.64</td>
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</tbody>
</table>

<table>
<thead>
<tr>
<th>n = 5</th>
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<th>( c = 5 )</th>
<th>( c = 7 )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Profit</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \theta = 0 )</td>
<td>0.18</td>
<td>0.22</td>
<td>0.34</td>
</tr>
<tr>
<td>( \theta = 1 )</td>
<td>0.28</td>
<td>0.29</td>
<td>0.60</td>
</tr>
<tr>
<td>( \theta = 2 )</td>
<td>0.38</td>
<td>0.19</td>
<td>0.71</td>
</tr>
<tr>
<td>Waste</td>
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<td></td>
<td></td>
</tr>
<tr>
<td>( \theta = 0 )</td>
<td>-10.21</td>
<td>-11.91</td>
<td>-22.75</td>
</tr>
<tr>
<td>( \theta = 1 )</td>
<td>-10.82</td>
<td>-11.46</td>
<td>-23.32</td>
</tr>
<tr>
<td>( \theta = 2 )</td>
<td>-11.89</td>
<td>NaN</td>
<td>-24.20</td>
</tr>
</tbody>
</table>

*Note. \( T = 15, r = 10, \alpha = 0.99, D = \text{Uniform}[0, 2] \). In the cases with “NaN”, the waste under no transshipment is close to zero.*
demand. If outlet 2 stocks a large amount of new inventory, which would make it difficult to sell old inventory, then the old inventory should be rotated to outlet 1. The following theorem is a generalization of Theorems 3 and 4.

**Theorem 9** Suppose that $D^2 \geq_{st} D^1$. Then at least one of the following statements must be true.

(i) $z^1_1 + y^1 < \bar{y}^2$;

(ii) Either $z^1_1$ or $\bar{y}^2$ is zero.

When Theorem 9 (i) does not hold (i.e., $z^1_1 + y^1 \geq \bar{y}^2$), then as much of the old inventory (if any) in the system as needed at outlet 2 should be sent there. It is easier to sell the old inventory at outlet 2 because of the greater demand and lower level of new inventory there. In this case, if $z^1_1 = 0$, then $\bar{y}^1 \geq \bar{y}^2$. If $\bar{y}^2 = 0$, then $z^2_1 \geq z^1_1$; that is, most of the old inventory is stocked at outlet 2.

7 Conclusions

In this study, we explore the idea of transshipment in the context of retailing of perishable goods. Whether or not transshipment is worth implementing depends on its benefits and costs. The practice can increase profit by as much as several percentage points when the variable cost of products is high. Its benefit in reducing waste can be a lot more substantial, and the benefit is highest also when the variable cost of products is high. Without a rigorous analysis like ours, such level of clarity is impossible. And as long as transshipment can be efficiently integrated into the regular replenishment process, the additional variable costs should be small. Implementation of transshipment may require changes in the business process, IT systems, and employee training upfront. These are all fixed costs, which do not affect our analysis and conclusions. Any retailers interested in the idea can weigh the benefits against the costs upfront and make an informed decision.

One possible challenge of implementing transshipment, as we mentioned earlier, stems from the misaligned incentive of store managers. This is not a major problem in convenience store chains (i.e., 7-Eleven Hong Kong) where one owner owns several outlets in close proximity and replenishment is completely centralized. In the extreme case, the incentive issue will go away if the stores are no longer managed by people. Unmanned stores such as Amazon Go in Seattle
and BingoBox in Shanghai have generated a lot of discussion (Dastin 2018 and Soo 2017). It would be fascinating to experiment transshipment in those stores.

Waste is a universal problem. Intense discussions have been made and government regulations implemented to increase the cost of waste disposal or even ban food waste in landfills. While waste will obviously be reduced if the cost of waste disposal is increased, it is a challenge to strike the right balance between the need to reduce waste and the business and consumer interests. Helping retailers to improve their operations, however, may boost their bottom lines and at the same time reduce waste. This is an area where operations researchers can do more.

Appendix

Proof of Lemma 1: Since the state variable $x_j$ appears in the linear term $s \sum_{j=1}^{n-1} x_j$ and in the constraint $z_1^j + z_2^j \leq x_j$, it is easy to see that $v_t(x + \delta e_j) - v_t(x) \geq s \delta$. This result shows that optimal profit is higher when initial inventory is higher.

(i) The proof is achieved by induction. The result obviously holds for $v_{T+1}(x)$. Suppose $v_{t+1}(x + \delta e_i) \leq v_{t+1}(x + \delta e_{i+1})$ for $1 \leq i \leq n - 2$. We first show that $v_t(x + \delta e_1) \leq v_t(x + \delta e_2)$.

Let $(\bar{z}^1, \bar{q}^1)$ and $(\bar{z}^2, \bar{q}^2)$ be the optimal solution for the state $x + \delta e_1$. If both $\bar{z}_1^1$ and $\bar{z}_1^2$ equal zero, then $(\bar{z}^1, \bar{q}^1)$ and $(\bar{z}^2, \bar{q}^2)$ are feasible for the state $x + \delta e_2$, and hence obviously $v_t(x + \delta e_2) \geq v_t(x + \delta e_1)$. Suppose either $\bar{z}_1^1$ or $\bar{z}_1^2$ is positive, and without loss of generality, let us assume $\bar{z}_1^1 > 0$. We can construct a new policy $(z^1, q^1)$ and $(z^2, q^2)$ where $(z^2, q^2) = (\bar{z}^2, \bar{q}^2)$, $z_1^1 = \bar{z}_1^1 - \delta$, $z_2^1 = \bar{z}_2^1 + \delta$, $z_j^j = \bar{z}_j^j$ and $q^1 = \bar{q}^1$ for $j \geq 3$. This new policy is feasible for $x + \delta e_2$, and

$$v_t(x + \delta e_2) - v_t(x + \delta e_1) \geq \theta \mathbb{E}(\bar{q}^1 + \sum_{j=2}^{n-1} \bar{z}_j^1 + \delta - D^1)^+ - \theta \mathbb{E}(\bar{q}^1 + \sum_{j=2}^{n-1} \bar{z}_j^1 - D^1)^+ + \Delta$$

where $\Delta = \alpha \mathbb{E}_{t+1}(\sum_{i=1}^{2} Y^i(q^i, z^i, D^i)) - \alpha \mathbb{E}_{t+1}(\sum_{i=1}^{2} Y^i(\bar{q}^i, \bar{z}^i, D^i))$ is the additional profit from future periods from using the new policy. The second inequality holds since the new policy leads to a higher initial inventory with a one-period lifetime in the next period, and hence the additional profit from future periods $\Delta$ is positive.

To show that $v_t(x + \delta e_i) \leq v_t(x + \delta e_{i+1})$, let $(\bar{z}^1, \bar{q}^1)$ and $(\bar{z}^2, \bar{q}^2)$ be the optimal solution for the state $x + \delta e_i$. If both $\bar{z}_1^1$ and $\bar{z}_1^2$ equal zero, then $(\bar{z}^1, \bar{q}^1)$ and $(\bar{z}^2, \bar{q}^2)$ are feasible for the state $x + \delta e_{i+1}$, and hence obviously $v_t(x + \delta e_{i+1}) \geq v_t(x + \delta e_i)$. If either $\bar{z}_1^1$ or $\bar{z}_1^2$ is positive, we can similarly construct a new policy $(z^1, q^1)$ and $(z^2, q^2)$ where $(z^2, q^2) = (\bar{z}^2, \bar{q}^2)$, $z_j^1 = \bar{z}_j^1 - \delta$, and in
The result follows because the derivative is decreasing in $q^1$ and $q^1 = q^1$ for $j < i$ and $j > i + 1$. This new policy is feasible for $x + \delta e_{i+1}$, and it leads to a fresher initial inventory in the next period. Hence by the induction hypothesis, we have $v_i(x + \delta e_{i+1}) \geq v_i(x + \delta e_i)$.

(ii) Let $(\bar{z}^1, \bar{q}^1)$ and $(\bar{z}^2, \bar{q}^2)$ be the optimal solution for the state $x + \delta e_{n-1}$. If both $\bar{z}^1_{n-1}$ and $\bar{z}^2_{n-1}$ equal zero, then $(\bar{z}^1, \bar{q}^1)$ and $(\bar{z}^2, \bar{q}^2)$ are feasible for the state $x$, and hence

$$v_i(x + \delta e_{n-1}) - v_i(x) \leq s\delta \leq c\delta.$$ 

The first inequality holds because $v_i(x)$ is greater than or equal to the profit under any feasible policy. Otherwise, let us assume $\bar{z}^1_{n-1} > 0$. We can construct a new policy $(z^1, q^1)$ and $(z^2, q^2)$ where $(z^2, q^2) = (\bar{z}^2, \bar{q}^2)$, $z^1_{n-1} = \bar{z}^1_{n-1} - \delta$, $q^1 = \bar{q}^1 + \delta$, $z^1_j = \bar{z}^1_j$ and $q^1_j = \bar{q}^1_j$ for $j < n - 1$. The new policy is feasible for the state $x$, and

$$v_i(x + \delta e_{n-1}) - v_i(x) \leq s\delta - s\delta + c\delta + \Delta \leq c\delta,$$

where $-\Delta = \alpha E v_{t+1}(\sum_{i=1}^{n} Y^i(q^i, z^i, D^i)) - \alpha E v_{t+1}(\sum_{i=1}^{n} Y^i(q^i, \bar{z}^i, D^i))$ is the additional profit from future periods from using the new policy. The second inequality holds since the new policy leads to fresher initial inventories in the next period, and hence $-\Delta$ is non-negative by part (i).

(iii) The first-order derivative of $J_t(z^i, q^i)$ with respect to $z^1_i$ is

$$\frac{\partial J_t(z^i, q^i)}{\partial z^1_i} = (p - s) - (p + \theta)\Phi(q^i + \sum_{j=1}^{n-1} z^1_j).$$

The result follows because the derivative is decreasing in $q^i$ and $z^j_i$. Furthermore, we can see that $J_t(z^i, q^i)$ is submodular in $(z^1_i, q^i + \sum_{j=1}^{n-1} z^1_j)$ for any $l \geq 2$. □

**Proof of Theorem 1:** The proof is achieved by contradiction. Suppose for all optimal policies we have $\bar{q}^1 > 0$ or $\bar{q}^2 > 0$ or $\bar{z}^1_j > 0$ or $\bar{z}^2_j > 0$ for some $j < i$.

If $\bar{q}^1 > 0$, we let $\delta = \min(x_i - \bar{z}^1_i - \bar{z}^1_j, \bar{q}^1)$ > 0 and construct a new policy $z^1 = \bar{z}^1 + \delta e_i$, and $q^1 = \bar{q}^1 - \delta$. The new policy will lead to an immediate profit increase of $(c - s)\delta$. However, the new policy will result in less fresh initial inventory in the next period, but the loss is smaller than $\alpha(c - s)\delta$. Hence the new policy will increase the profit, which is a contradiction. Therefore, $\bar{q}^1 = 0$ and similarly we can show that $\bar{q}^2 = 0$.

If $\bar{z}^1_j > 0$ for some $j < i$, we let $\delta = \min(x_i - \bar{z}^1_i - \bar{z}^2_j, \bar{z}^1_j) > 0$ and construct a new policy $z^1 = \bar{z}^1 - \delta e_j + \delta e_i$, and $q^1 = \bar{q}^1$. The new policy will increase profit by $\theta(\mathbb{E}(\bar{q}^1 + \delta - D^1)^+ - \mathbb{E}(\bar{q}^1 - D^1)^+)$.
in the current period and will lead to fresher initial inventory in the next period, which is a contradiction. Thus, \( \bar{z}_j^1 = 0 \) and similarly we can show that \( \bar{z}_j^2 = 0 \). \( \square \)

**Proof of Theorem 2**: Suppose that \( \sum_{i=2}^{n-1} x_i \geq 2l_0 \). If there exists an \( i \geq 2 \) such that \( \bar{z}_i^1 + \bar{z}_i^2 < x_i \), then the result holds according to Theorem 1. Otherwise, we have \( \bar{z}_i^1 + \bar{z}_i^2 = x_i \) for all \( 2 \leq i \leq n - 1 \). Because \( \sum_{i=2}^{n-1} x_i \geq 2l_0 \), we have \( \sum_{i=2}^{n-1} \bar{z}_i^1 + \sum_{i=2}^{n-1} \bar{z}_i^2 \geq 2l_0 \), which means either \( \sum_{i=2}^{n-1} \bar{z}_i^1 \geq l_0 \) or \( \sum_{i=2}^{n-1} \bar{z}_i^2 \geq l_0 \). The derivative of \( J_t(z^i, q^i) \) with respect to \( z_1^i \) is

\[
(p + \bar{y})(\Phi(l_0) - \Phi(z_1^i + \sum_{j=2}^{n-1} z_j^i + q^i)),
\]

which is negative if \( \sum_{j=2}^{n-1} z_j^i \geq l_0 \). Therefore, when \( \sum_{j=2}^{n-1} z_j^i \geq l_0 \), either \( \bar{z}_1^i \) or \( \bar{z}_2^i \) is zero. \( \square \)

**Proof of Lemma 2**: (i) For any \( D^i \),

\[
(z_1^i - (D^i - y^i)^+) = (z_1^i + y^i - D^i)^+ - (y^i - D^i)^+,
\]

and

\[
\min(D^i, z_1^i + y^i) = z_1^i + y^i - (z_1^i + y^i - D^i)^+.
\]

So

\[
J_t(z_1^i, y^i) = (p - s)z_1^i + (p - c)y^i - (p + \bar{y})E(z_1^i + y^i - D^i)^+ + (\bar{y} + v)E(y^i - D^i)^+.
\]

The first, second and last terms depend on only one variable, and they are hence submodular. The third is a concave function of \( z_1^i + y^i \) and is therefore submodular in \( (z_1^i, y^i) \) (Lemma 2.6.2, Topkis 1998). The result follows because the sum of submodular functions is still submodular.

(ii) Let

\[
f^i(y) = (p - s)z_1^i + (p - c)y + (p - c)\mu^i - (p + \bar{y})(z_1^i + y)^+ + (\bar{y} + v)y^+.
\]

Then \( J_t(z_1^i, y^i) = Ef^i(y^i - D^i) \). It is easy to show that \( f^i(y) \) is quasiconcave; it is first increasing, then decreasing, and finally decreasing but with a more gentle slope. The result hence follows. \( \square \)

**Proof of Theorem 3**: The proof is achieved by contradiction. Suppose that in all optimal policies, \( \bar{z}_1^1 > 0, \bar{z}_2^1 > 0, \bar{y}^1 > 0, \) and \( \bar{y}^2 > 0 \). Without loss of generality, assume \( \bar{y}_1^1 \geq \bar{y}_2^1 \).

Let \( \delta = \min(\bar{z}_1^1, \bar{z}_2^1) \) and we construct a new policy:

\[
z_1^i = \bar{z}_1^i - \delta, \quad z_2^i = \bar{z}_2^i + \min(\delta, \bar{z}_2^2), \quad y_1^i = \bar{y}_1^1 + \delta, \quad z_1^1 = \bar{z}_1^2 + \delta, \quad z_2^2 = \bar{z}_2^2 - \min(\delta, \bar{z}_2^2), \quad y_2^2 = \bar{y}_2^2 - \delta.
\]
It is not difficult to show that the new policy is still feasible and either $z_1^1$ or $y^2$ is zero and $z_1^1, z_2^1, y^1, z_1^2, z_2^2$, and $y^2$ are all nonnegative. The objective function $J_t$ can be written as

$$J_t(z_1^1, y^1) = -sz_1^1 - cy^1 + pE\min(D^1, z_1^1 + y^1) - \theta E(z_1^1 + y^1 - D^1)^+ + (\theta + \alpha v)E(y^1 - D^1)^+.$$ 

The expected profit under the new policy minus that under the optimal policy is

$$(\theta + \alpha v)[E(\bar{y}^1 + \delta - D^1)^+ + E(y^2 - \delta - D^2)^+ - E(\bar{y}^1 - D^1)^+ - E(y^2 - D^2)^+]$$

Because the function $E(x - D^1)^+$ is a convex function, the above expression is positive. This means that the new policy is also optimal, which is a contradiction.

**Proof of Theorem 4:** (i) When $\bar{z}_1^1 + \bar{z}_1^2 > 0$, because at least one of $(\bar{z}_1^1, \bar{z}_1^2, \bar{y}^1, \bar{y}^2)$ must be zero (Theorem 3), consider the following two cases:

1. At least one of $(\bar{z}_1^1, \bar{z}_1^2)$ is zero. Because $\bar{z}_1^1 + \bar{z}_1^2 > 0$, if one of them is zero, the other must be strictly positive. Therefore, the optimal solution is asymmetric.

2. At least one of $(\bar{y}^1, \bar{y}^2)$ is zero. If both of them are zero, then according to Theorem 1, we have $\bar{z}_1^1 = \bar{z}_1^2 = 0$, which is a contradiction. Therefore, in this case, one of them is zero and the other is strictly positive and the optimal solution is asymmetric.

(ii) When $\bar{z}_1^1 + \bar{z}_1^2 = 0$,

$$v_t(x_1, x_2) = s(x_1 + x_2) + \max_{\bar{z}_1^1 + \bar{z}_1^2 \leq x_2, \bar{z}_1^1 \geq 0} \{(c - s)(\bar{z}_1^1 + \bar{z}_1^2) + J_t(0, \bar{z}_1^1) + J_t(0, \bar{z}_1^2)\}.$$ 

The function $(c - s)z + J_t(0, z)$ is concave in $z$. Suppose there is an optimal solution such that $\bar{z}_1^1 \neq \bar{z}_1^2$. Then, the symmetric solution $(\bar{z}_1^1 + \bar{z}_1^2, \bar{z}_1^1 + \bar{z}_1^2)$ is also an optimal solution. □

Throughout the rest of the appendix, we use the notations $\bar{z}_k^j(x_1, x_2|K_i)$ and $\bar{y}_k^j(x_1, x_2|K_i)$ to denote the optimal solutions to the optimization problem $K_i$ in (4). The following lemma characterizes the monotonicity of the optimal solution to each of the three maximization problems in (4).

**Lemma 3**

(i) The function $\bar{z}_1^1(x_1, x_2|K_1)$ is increasing in $x_1$ and decreasing in $x_2$, and $\bar{y}_1^1(x_1, x_2|K_1)$ is decreasing in $x_1$ and increasing in $x_2$;

(ii) The function $\bar{z}_1^2(x_1, x_2|K_2)$ is increasing in $x_1$ and decreasing in $x_2$, and $\bar{y}_2^2(x_1, x_2|K_2)$ is decreasing in $x_1$ and increasing in $x_2$.

(iii) If $x_2 < 2u_0$, $K_3(x_1, x_2) \leq K_2(x_1, x_2)$.
Proof of Lemma 3: (i) We first look at the optimization problem $K_1$. Let $\hat{y}^1 = -y^1, \hat{x}_2 = -x_2$ and $x_1 - z^2_1 = \hat{z}^2_1$. Then the objective function is supermodular in $(z^1_1, \hat{z}^2_1, \hat{y}^1, x_1, \hat{x}_2)$ and the constraint set forms a lattice. Therefore, $\hat{z}^1_1(x_1, x_2|K_1)$ is increasing in $x_1$ and decreasing in $x_2$, and $\hat{y}^1(x_1, x_2|K_1)$ is decreasing in $x_1$ and increasing in $x_2$.

(ii) We next look at the optimization problem $K_2$. Let $\hat{y}^1 = x_2 - y^1, \hat{x}_2 = -x_1$ and $\hat{z}^2_1 = -z^2_1$. Then the objective function is supermodular in $(\hat{z}^2_1, \hat{y}^1, y_2, \hat{x}_1, x_2)$ and the constraint set forms a lattice. Therefore, $\hat{y}^2(x_1, x_2|K_2)$ is decreasing in $x_1$ and increasing in $x_2$, and $\hat{z}^2_1(x_1, x_2|K_2)$ is increasing in $x_1$ and decreasing in $x_2$.

(iii) Note first that $u_0 = \arg \max_{z \geq 0} \{(c - s)z + J_t(0, z)\}$. When $x_2 < 2u_0$, the optimal solution to $K_3$ must be a boundary solution; that is, $\hat{z}^1_2(x_1, x_2|K_3) + \hat{z}^2_2(x_1, x_2|K_3) = x_2$. It is easy to see that the optimal solution to $K_3$ is a feasible but not necessarily optimal solution to $K_2$. □

Proof of Theorem 5: The proof of the theorem is based on the optimization problem (4).

(i) For $x_2 \geq 2u_0$, if we can show that $K_1 \leq K_2 \leq K_3$, then the result follows. We can simplify the optimization problem $K_1$ by sequentially optimizing the objective function with respect to each decision variable, first $z^2_1$, then $z^1_1$, and finally $y^1$. Note that

$$\frac{\partial J_t(z, y)}{\partial z} = p - s - (p + \theta)\Phi(z + y) = (p + \theta)(\Phi(l_0) - \Phi(z + y)).$$

We obtain the following two cases after optimizing over $z^2_1$.

Case 1: If $x_1 \leq l_0$,

$$K_1 - (c - s)x_2 = \max_{0 \leq z^2_1 \leq x_1 - z^1_1} \{J_t(z^1_1, y^1) + J_t(z^2_1, 0)\} = \max_{y^1 \geq 0, 0 \leq z^1_1 \leq x_1} \{J_t(z^1_1, y^1) + J_t(x_1 - z^1_1, 0)\}.$$ 

Based on the following derivative,

$$\frac{\partial\{J_t(z^1_1, y^1) + J_t(x_1 - z^1_1, 0)\}}{\partial z^1_1} = (p + \theta)(\Phi(x_1 - z^1_1) - \Phi(z^1_1 + y^1)),$$

we know that $\max_{z^1_1 \geq 0} J_t(z^1_1, y^1) + J_t(x_1 - z^1_1, 0) = (x_1 - y^1)^+ / 2$. There are two sub-cases after optimizing over $z^1_1$.

Case 1(i): If $x_2 \leq x_1$,

$$K_1 - (c - s)x_2 = \max_{y^1 \geq x_1} \{\max_{x_2 \leq y^1 \leq x_1} \{J_t(0, y^1) + J_t(x_1, 0)\}, \max_{x_2 \leq y^1 \leq x_1} \{J_t(\frac{x_1 - y^1}{2}, y^1) + J_t(\frac{x_1 + y^1}{2}, 0)\}\}.$$
Case 1(ii) If $x_2 > x_1$,

$$K_1 - (c - s)x_2 = \max_{y^1 \geq x_2} \{J_t(0, y^1) + J_t(x_1, 0)\}.$$ 

Case 2: If $x_1 > l_0$,

$$K_1 - (c - s)x_2 = \max_{0 \leq z_1^1 \leq x_1 - z_1^1, y^1 \geq x_2, z_1^1 \geq 0} \{J_t(z_1^1, y^1) + J_t(z_1^2, 0)\}$$

$$= \max_{x_1 - z_1^1 \leq l_0, y^1 \geq x_2, z_1^1 \geq 0} \{J_t(z_1^1, y^1) + J_t(l_0, 0)\},$$

$$= \max_{x_1 - z_1^1 < l_0, y^1 \geq x_2, z_1^1 \geq 0} \{J_t(z_1^1, y^1) + J_t(x_1 - z_1^1, 0)\}.$$ 

Optimizing over $z_1^1$, we have the following three sub-cases.

Case 2(i): If $x_2 \leq 2l_0 - x_1$,

$$K_1 - (c - s)x_2 = \max_{2l_0 - x_1 \leq y^1 \leq l_0, x_2 \leq y^1 \leq 2l_0 - x_1} \{J_t(l_0 - y^1, y^1) + J_t(l_0, 0)\},$$

$$\max_{y^1 \geq 2l_0 - x_1} \{J_t(x_1 - l_0, y^1) + J_t(l_0, 0)\},$$

$$= \max_{x_2 \leq y^1 \leq 2l_0 - x_1} \{J_t(x_1 - l_0, y^1) + J_t(l_0, 0)\}.$$ 

The second equality holds since $J_t(l_0 - y^1, y^1)$ is convex in $y^1$ and its derivative is

$$\frac{\partial}{\partial y^1} \{J_t(l_0 - y^1, y^1)\} = s - c + (\theta + \alpha v)\Phi(y^1).$$

Case 2(ii): If $2l_0 - x_1 \leq x_2 \leq l_0$,

$$K_1 - (c - s)x_2 = \max_{x_2 \leq y^1 \leq l_0, y^1 \geq x_2} \{J_t(x_1 - l_0, y^1) + J_t(l_0, 0)\}.$$ 

Case 2(iii): If $x_2 \geq l_0$,

$$K_1 - (c - s)x_2 = \max_{y^1 \geq x_2} \{J_t(0, y^1) + J_t(l_0, 0)\}.$$ 

From the above discussion, we know that for $x_2 \geq 2u_0 \geq 2l_0$, either we have Case 2(iii) where

$$K_1 - (c - s)x_2 = \max_{y^1 \geq x_2} \{J_t(0, y^1) + J_t(l_0, 0)\},$$

or we have Case 1(ii) where

$$K_1 - (c - s)x_2 = \max_{y^1 \geq x_2} \{J_t(0, y^1) + J_t(x_1, 0)\}.$$
Since $J_t(x_1 - l_0, y^1) \leq J_t(x_1 - l_0, y^1)$ for any $y^1 \geq x_2 \geq 2u_0$, we have $z_t^1(x_1, x_2 | K_1) = 0$ and $K_1 \leq K_2$.

We can similarly simplify the optimization problem $K_2$ by sequentially optimizing the objective function with respect to each decision variable, first $y^1$, then $z_t^2$, and finally $y^2$. Note that
\[
\frac{\partial J_t(0, y)}{\partial y} = p - c - (p - \alpha v)\Phi(y).
\]

There are two cases after optimizing over $y^1$.

**Case 1:** If $x_2 \leq u$,
\[
K_2 - (c - s)x_2 = \max_{0 \leq z_t^1 \leq x_1, y^2 \geq 0} \{J_t(0, y^1) + J_t(z_t^2, y^2)\}
\]
\[
= \max_{0 \leq z_t^1 \leq x_1, y^2 \geq 0} \{J_t(0, u) + J_t(z_t^2, y^2)\}.
\]

Optimizing over $z_t^2$, we further obtain two sub-cases.

**Case 1(i):** If $x_1 \geq l_0$,
\[
K_2 - (c - s)x_2 = \max_{0 \leq z_t^1 \leq x_1, y^2 \geq 0} \{J_t(0, u) + J_t(z_t^2, y^2)\}
\]
\[
= \max \left\{ \max_{0 \leq y^2 \leq l_0} \{J_t(0, u) + J_t(l_0 - y^2, y^2)\}, \max_{0 \leq y^2 \leq l_0 - x_1} \{J_t(0, u) + J_t(x_1, y^2)\} \right\}.
\]

**Case 1(ii):** If $x_1 \leq l_0$,
\[
K_2 - (c - s)x_2 = \max_{0 \leq z_t^1 \leq x_1, y^2 \geq 0} \{J_t(0, u) + J_t(z_t^2, y^2)\}
\]
\[
= \max \left\{ \max_{l_0 - x_1 \leq y^2 \leq l_0} \{J_t(0, u) + J_t(l_0 - y^2, y^2)\}, \max_{0 \leq y^2 \leq l_0 - x_1} \{J_t(0, u) + J_t(x_1, y^2)\} \right\}.
\]

**Case 2:** If $x_2 \geq u$,
\[
K_2 - (c - s)x_2 = \max_{0 \leq z_t^1 \leq x_1, y^2 \geq 0} \{J_t(0, y^1) + J_t(z_t^2, y^2)\}
\]
\[
= \max \left\{ \max_{0 \leq z_t^1 \leq x_1, 0 \leq y^2 \leq x_2 - u} \{J_t(0, x_2 - y^2) + J_t(z_t^2, y^2)\}, \max_{0 \leq y^2 \leq x_2 - u} \{J_t(0, u) + J_t(z_t^2, y^2)\} \right\}.
\]

Optimizing over $z_t^2$, we obtain the following five sub-cases.

**Case 2(i):** If $x_1 \geq l_0$ and $x_2 - u \geq l_0$,
\[
K_2 - (c - s)x_2 = \max \left\{ \max_{0 \leq z_t^1 \leq x_1, 0 \leq y^2 \leq x_2 - u} \{J_t(0, x_2 - y^2) + J_t(z_t^2, y^2)\}, \max_{0 \leq z_t^1 \leq x_1, 0 \leq y^2 \leq x_2 - u} \{J_t(0, u) + J_t(z_t^2, y^2)\} \right\}
\]
\[
= \max \left\{ \max_{0 \leq y^2 \leq l_0} \{J_t(0, x_2 - y^2) + J_t(l_0 - y^2, y^2)\}, \max_{0 \leq y^2 \leq l_0 - x_1} \{J_t(0, x_2 - y^2) + J_t(0, y^2)\} \right\}.
\]
Case 2(ii): If $x_1 \geq l_0$ and $x_2 - u \leq l_0$

$$K_2 - (c - s)x_2 = \max \left\{ \max_{0 \leq y^2 \leq x_2 - u} \left\{ J_t(0, x_2 - y^2) + J_t(z_1^2, y^2) \right\}, \max_{0 \leq z_1^2 \leq x_1, 0 \leq y^2 \leq x_2 - u} \left\{ J_t(0, u) + J_t(z_1^2, y^2) \right\} \right\}$$

$$= \max \left\{ \max_{0 \leq y^2 \leq x_2 - u} \left\{ J_t(0, x_2 - y^2) + J_t(l_0 - y^2, y^2) \right\}, \max_{x_2 - u \leq y^2 \leq l_0 - x_1} \left\{ J_t(0, u) + J_t(l_0 - y^2, y^2) \right\}, \max_{y^2 \geq 0} \left\{ J_t(0, u) + J_t(0, y^2) \right\} \right\}$$

Case 2(iii): If $x_1 \leq l_0$ and $l_0 - x_1 \geq x_2 - u$

$$K_2 - (c - s)x_2 = \max \left\{ \max_{0 \leq y^2 \leq x_2 - u} \left\{ J_t(0, x_2 - y^2) + J_t(x_1, y^2) \right\}, \max_{l_0 - x_1 \leq y^2 \leq l_0} \left\{ J_t(0, u) + J_t(l_0 - y^2, y^2) \right\}, \max_{x_2 - u \leq y^2 \leq l_0 - x_1} \left\{ J_t(0, u) + J_t(x_1, y^2) \right\}, \max_{y^2 \geq 0} \left\{ J_t(0, u) + J_t(0, max\{u, l_0\}) \right\} \right\}$$

Case 2(iv): If $x_1 \leq l_0$ and $l_0 - x_1 \leq x_2 - u \leq l_0$

$$K_2 - (c - s)x_2 = \max \left\{ \max_{l_0 - x_1 \leq y^2 \leq x_2 - u} \left\{ J_t(0, x_2 - y^2) + J_t(l_0 - y^2, y^2) \right\}, \max_{0 \leq y^2 \leq l_0 - x_1} \left\{ J_t(0, x_2 - y^2) + J_t(x_1, y^2) \right\}, \max_{x_2 - u \leq y^2 \leq l_0} \left\{ J_t(0, u) + J_t(l_0 - y^2, y^2) \right\}, \max_{y^2 \geq 0} \left\{ J_t(0, u) + J_t(0, max\{u, l_0\}) \right\} \right\}$$

Case 2(v): If $x_1 \leq l_0$ and $x_2 - u \geq l_0,$

$$K_2 - (c - s)x_2 = \max \left\{ \max_{l_0 - x_1 \leq y^2 \leq l_0} \left\{ J_t(0, x_2 - y^2) + J_t(l_0 - y^2, y^2) \right\}, \max_{0 \leq y^2 \leq l_0 - x_1} \left\{ J_t(0, x_2 - y^2) + J_t(x_1, y^2) \right\}, \max_{0 \leq y^2 \leq l_0 - x_1} \left\{ J_t(0, x_2 - y^2) + J_t(x_1, y^2) \right\} \right\}$$

From the above discussion, we know that for $x_2 \geq 2u_0,$ either we have Case 2(i) where

$$K_2 - (c - s)x_2 = \max \left\{ \max_{0 \leq y^2 \leq l_0} \left\{ J_t(0, x_2 - y^2) + J_t(l_0 - y^2, y^2) \right\}, \max_{l_0 \leq y^2 \leq x_2 - u} \left\{ J_t(0, u) + J_t(0, y^2) \right\} \right\}.$$ 

or we have Case 2(v) where

$$K_2 - (c - s)x_2 = \max \left\{ \max_{l_0 - x_1 \leq y^2 \leq l_0} \left\{ J_t(0, x_2 - y^2) + J_t(l_0 - y^2, y^2) \right\}, \max_{0 \leq y^2 \leq l_0 - x_1} \left\{ J_t(0, x_2 - y^2) + J_t(x_1, y^2) \right\}, \max_{l_0 \leq y^2 \leq x_2 - u} \left\{ J_t(0, x_2 - y^2) + J_t(0, y^2) \right\} \right\}.$$
Because for $0 \leq y^2 \leq l_0 - x_1$,
\[
\frac{\partial}{\partial y^2} (J_t(0, x_2 - y^2) + J_t(x_1, y^2)) = (p - \alpha v)\Phi(x_2 - y^2) - (p + \theta)\Phi(x_1 + y^2) + (\theta + \alpha v)\Phi(y^2)
\]
\[
\geq (p - \alpha v)\Phi(x_2 - y^2) - (p + \theta)\Phi(x_1 + y^2)
\]
\[
\geq (p - \alpha v)\Phi(x_2 + x_1 - l_0) - (p + \theta)\Phi(l_0)
\]
\[
\geq 0
\]
and for $0 \leq y^2 \leq l_0$,
\[
\frac{\partial}{\partial y^2} (J_t(0, x_2 - y^2) + J_t(l_0 - y^2, y^2)) = -(p - s) + (p - \alpha v)\Phi(x_2 - y^2) + (\theta + \alpha v)\Phi(y^2)
\]
\[
\geq -(p - s) + (p - \alpha v)\Phi(x_2 - l_0)
\]
\[
\geq 0,
\]
we can simplify $K_2$ as
\[
K_2 - (c - s)x_2 = \max_{l_0 \leq y^2 \leq x_2 - u} \{J_t(0, x_2 - y^2) + J_t(0, y^2)\}
\]
\[
= J_t(0, x_2/2) + J_t(0, x_2/2).
\]
Thus, we have $K_2 \leq K_3$.

(ii) According to Lemma 3(i) and (ii), we can define $A_1(x_2) = \max\{x_1 : \exists^1(x_1, x_2 | K_1) = 0\}$ and $A_2(x_2) = \max\{x_1 : \exists^2(x_1, x_2 | K_2) = 0\}$. Both $A_1(x_2)$ and $A_2(x_2)$ are increasing in $x_2$. Let $A(x_2) = \min\{A_1(x_2), A_2(x_2)\}$, then it is easy to see that it is optimal clear all old inventory if $x_1 \leq A(x_2)$. □

We introduce the following result which will be used in later proofs.

**Lemma 4** If $u \leq l_0$, then $J_t(l_0, 0) \geq J_t(0, u)$.

**Proof of Lemma 4:** The inequality $u \leq l_0$ is equivalent to $\frac{c - s}{\theta + \alpha v} \geq \Phi(l_0)$. Because $\frac{\partial J_t(x, 0)}{\partial x} \geq \frac{\partial J_t(0, x)}{\partial x}$ for $\Phi(x) \leq \frac{c - s}{\theta + \alpha v}$, we have $J_t(l_0, 0) \geq J_t(0, u)$. □

**Proof of Theorems 6, 7, and 8:** Since Theorems 6, 7, and 8 focus on different aspects of the optimal policy, here we provide a unified proof by first characterizing the optimal policy under different states and then defining appropriate switching curves $B(x_1)$ and $C(x_1)$.

Let $O(x_1) = \arg\max_{y^1 \geq 0} J_t(x_1, y^1)$ and $l = \Phi^{-1}(\frac{c - s}{\theta + \alpha v})$. It is easy to show that $O(x_1)$ is decreasing and $O(x_1) = 0$ for $x_1 \geq l$. Moreover, $O(x_1) \leq u$. Based on the values of the parameters, the proof below is divided into three parts. In Part I, $u \leq l_0$. In Part II, $u > l_0$ and $J_t(l_0, 0) \leq J_t(0, u)$. In Part III, $u > l_0$ and $J_t(l_0, 0) > J_t(0, u)$.  

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Part I: From the proof of Theorem 5, we can characterize the optimal policy for the optimization problem $K_1$ as follows.

Case 1(i): If $x_1 \leq l_0$ and $x_2 \leq x_1$, 

$$K_1 - (c - s)x_2 = \max \{ \max_{y^1 \geq x_2} \{ J_t(0, y^1) + J_t(x_1, 0) \}, \max_{x_2 \leq y^1 \leq x_1} \{ J_t(\frac{x_1 - y^1}{2}, y^1) + J_t(\frac{x_1 + y^1}{2}, 0) \} \}. $$

In this case, when $x_1 \leq u$, $K_1(x_1, x_2) = (c - s)x_2 + J_t(0, u) + J_t(x_1, 0)$; and when $x_1 \geq u$, $K_1$ can be simplified as:

$$K_1(x_1, x_2) = (c - s)x_2 + \max_{x_2 \leq y^1 \leq x_1} \{ J_t(\frac{x_1 - y^1}{2}, y^1) + J_t(\frac{x_1 + y^1}{2}, 0) \}. $$

Note that here the optimal $y^1 < x_1$.

Case 1(ii): If $x_1 \leq l_0$ and $x_2 > x_1$,

$$K_1 - (c - s)x_2 = \max_{y^1 \geq x_2} \{ J_t(0, y^1) + J_t(x_1, 0) \} = J_t(0, \max\{x_2, u\}) + J_t(x_1, 0).$$

Case 2(i): If $x_1 \geq l_0$ and $x_2 \leq 2l_0 - x_1$,

$$K_1 - (c - s)x_2 = \max\{ J_t(0, l_0) + J_t(l_0, 0), \max_{y^1 \geq x_2} \{ J_t(x_1 - l_0, y^1) + J_t(l_0, 0) \}, \max_{x_2 \leq y^1 \leq 2l_0 - x_1} \{ J_t(x_1 - l_0, y^1) + J_t(\frac{x_1 + y^1}{2}, 0) \} \} = J_t(x_1 - l_0, \max\{x_2, O(x_1 - l_0)\}) + J_t(l_0, 0).$$

Case 2(ii): If $x_1 \geq l_0$ and $2l_0 - x_1 \leq x_2 \leq l_0$,

$$K_1 - (c - s)x_2 = \max\{ \max_{x_2 \leq y^1 \leq l_0} \{ J_t(l_0 - y^1, y^1) + J_t(l_0, 0) \}, \max_{y^1 \geq x_2} \{ J_t(0, y^1) + J_t(l_0, 0) \}, \max_{y^1 \geq x_2} \{ J_t(x_1 - l_0, y^1) + J_t(l_0, 0) \} \} = J_t(l_0 - x_2, x_2) + J_t(l_0, 0).$$

Case 2(iii): If $x_1 \geq l_0$ and $x_2 \geq l_0$,

$$K_1 - (c - s)x_2 = \max\{ \max_{y^1 \geq x_2} \{ J_t(0, y^1) + J_t(l_0, 0) \}, \max_{y^1 \geq x_2} \{ J_t(x_1 - l_0, y^1) + J_t(l_0, 0) \} \} = J_t(0, x_2) + J_t(l_0, 0).$$

Similarly, we can characterize the optimal policy for the optimization problem $K_2$.

Case 1(i): If $x_2 \leq u$ and $x_1 \geq l_0$,

$$K_2 - (c - s)x_2 = \max_{0 \leq z_1^2 \leq x_1, y^2 \geq 0} \{ J_t(0, u) + J_t(z_1^2, y^2) \} = \max\{ \max_{0 \leq y^2 \leq l_0} \{ J_t(0, u) + J_t(l_0 - y^2, y^2) \}, \max_{y^2 \geq l_0} \{ J_t(0, u) + J_t(0, y^2) \} \} = J_t(0, u) + J_t(l_0, 0).$$
Case 1(ii): If \( x_2 \leq u \) and \( x_1 \leq l_0 \),

\[
K_2 - (c-s)x_2 = \max_{0 \leq t^1 \leq x_1, y^2 \geq 0} \left\{ J_t(0, u) + J_t(z^2_1, y^2) \right\}
\]

\[
= \max\left\{ \max_{0 \leq y^2 \leq l_0} \left\{ J_t(0, u) + J_t(l_0 - y^2, y^2) \right\}, \max_{0 \leq y^2 \leq l_0 - x_1} \left\{ J_t(0, u) + J_t(x_1, y^2) \right\} \right\}
\]

\[
= J_t(0, u) + J_t(x_1, O(x_1)).
\]

Case 2(i): If \( x_1 \geq l_0 \) and \( x_2 - u \geq l_0 \),

\[
K_2 - (c-s)x_2 = \max\left\{ \max_{0 \leq t^1 \leq x_1, 0 \leq y^2 \leq x_2-u} \left\{ J_t(0, x_2 - y^2) + J_t(z^2_1, y^2) \right\}, \max_{0 \leq y^2 \leq l_0 - x_2-u} \left\{ J_t(0, x_2 - y^2) + J_t(0, y^2) \right\} \right\}
\]

\[
= J_t(0, x_2) + J_t(l_0, 0).
\]

Case 2(ii): If \( x_1 \geq l_0 \) and \( u \leq x_2 \leq u + l_0 \),

\[
K_2 - (c-s)x_2 = \max\left\{ \max_{0 \leq t^1 \leq x_1, 0 \leq y^2 \leq x_2-u} \left\{ J_t(0, x_2 - y^2) + J_t(z^2_1, y^2) \right\}, \max_{0 \leq y^2 \leq l_0 - x_2-u} \left\{ J_t(0, x_2 - y^2) + J_t(0, y^2) \right\} \right\}
\]

\[
= J_t(0, x_2) + J_t(l_0, 0).
\]

Case 2(iii): If \( x_1 \leq l_0 \) and \( l_0 - x_1 \geq x_2 - u \geq 0 \),

\[
K_2 - (c-s)x_2 = \max\left\{ \max_{0 \leq y^2 \leq x_2-u} \left\{ J_t(0, x_2 - y^2) + J_t(x_1, y^2) \right\}, \max_{l_0-x_1 \leq y^2 \leq l_0} \left\{ J_t(0, u) + J_t(l_0 - y^2, y^2) \right\}, \right\}
\]

\[
= \max\left\{ \max_{0 \leq y^2 \leq x_2-u} \left\{ J_t(0, x_2 - y^2) + J_t(x_1, y^2) \right\}, \max_{l_0-x_1 \leq y^2 \leq l_0} \left\{ J_t(0, u) + J_t(l_0 - y^2, y^2) \right\}, \right\}
\]

\[
= J_t(0, x_2) + J_t(l_0, O(x_1)).
\]

When \( u \leq l_0 \), we can show that \( O(x_1) \leq l_0 - x_1 \). Thus if \( x_2 - u \leq O(x_1) \), \( K_2(x_1, x_2) = (c-s)x_2 + J_t(0, u) + J_t(x_1, O(x_1)) \).

Case 2(iv): If \( x_1 \leq l_0 \) and \( l_0 - x_1 \leq x_2 - u \leq l_0 \),

\[
K_2 - (c-s)x_2 = \max\left\{ \max_{l_0-x_1 \leq y^2 \leq x_2-u} \left\{ J_t(0, x_2 - y^2) + J_t(l_0 - y^2, y^2) \right\}, \max_{l_0-x_1 \leq y^2 \leq l_0 - x_1} \left\{ J_t(0, x_2 - y^2) + J_t(0, y^2) \right\} \right\}
\]

\[
= \max\left\{ \max_{l_0-x_1 \leq y^2 \leq x_2-u} \left\{ J_t(0, x_2 - y^2) + J_t(l_0 - y^2, y^2) \right\}, \max_{0 \leq y^2 \leq l_0 - x_1} \left\{ J_t(0, x_2 - y^2) + J_t(x_1, y^2) \right\} \right\}
\]

\[
= J_t(0, u) + J_t(l_0 - y^2, y^2) \right\}.
\]

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Case 2(v): If \( x_1 \leq l_0 \) and \( x_2 - u \geq l_0 \),
\[
K_2 - (c-s)x_2 = \max \{ \max_{l_0 - x_1 \leq y^2 \leq l_0} \{ J_t(0, x_2 - y^2) + J_t(l_0 - y^2, y^2) \}, \max_{0 \leq y^2 \leq l_0 - x_1} \{ J_t(0, x_2 - y^2) + J_t(x_1, y^2) \} \}
\]
\[
= \max \{ \max_{l_0 - x_1 \leq y^2 \leq l_0} \{ J_t(0, x_2 - y^2) + J_t(l_0 - y^2, y^2) \}, \max_{y^2 \geq x_2 - u} \{ J_t(0, u) + J_t(0, y^2) \} \}
\]
\[
= \max \max \{ J_t(0, x_2 - y^2) + J_t(l_0 - y^2, y^2) \}, \max \{ J_t(0, x_2 - y^2) + J_t(x_1, y^2) \} \}
\]
\[
= \max \{ J_t(0, x_2 - y^2) + J_t(l_0 - y^2, y^2) \}, \max \{ J_t(0, x_2 - y^2) + J_t(x_1, y^2) \} \}
\]
For Cases 2(iii),(iv),(v), we know that if \( x_2 - u \geq O(x_1) \), for \( 0 \leq y^2 \leq x_2 - u \),
\[
\frac{\partial \{ J_t(0, x_2 - y^2) + J_t(x_1, y^2) \}}{\partial y^2} = (p - \alpha v)\Phi(x_2 - y^2) - (p + \theta)\Phi(x_1 + y^2) + (\theta + \alpha v)\Phi(y^2)
\]
\[
\geq (p - c) - (p + \theta)\Phi(x_1 + y^2) + (\theta + \alpha v)\Phi(y^2),
\]
which is greater than zero for \( y^2 \leq O(x_1) \), and hence \( \bar{y}^2(x_1, x_2|K_2) > 0 \) if \( x_1 \leq l_0 \). If \( x_1 \geq u \) and \( x_1 \geq x_2 \),
\[
\frac{\partial \{ J_t(0, x_2 - y^2) + J_t(x_1, y^2) \}}{\partial y^2} = (p - \alpha v)\Phi(x_2 - y^2) - (p + \theta)\Phi(x_1 + y^2) + (\theta + \alpha v)\Phi(y^2)
\]
\[
\leq (p - \alpha v)\Phi(x_2) - (p + \theta)\Phi(x_1) + (\theta + \alpha v)\Phi(x_2 - u)
\]
\[
\leq (\theta + \alpha v)\Phi(x_2 - u) - \Phi(x_1)
\]
\[
\leq 0,
\]
and
\[
\frac{\partial \{ J_t(0, x_2 - y^2) + J_t(l_0 - y^2, y^2) \}}{\partial y^2} = -(p - s) + (p - \alpha v)\Phi(x_2 - y^2) + (\theta + \alpha v)\Phi(y^2)
\]
\[
\leq -(p - s) + (p - \alpha v)\Phi(x_2 - x_1 - l_0) + (\theta + \alpha v)\Phi(x_2 - u)
\]
\[
\leq -(p - s) + (p + \theta)\Phi(x_2)
\]
\[
\leq 0,
\]
and hence \( \bar{y}^2(x_1, x_2|K_2) = 0 \).

From the above discussion, if we let
\[
C(x_1) = \begin{cases} 
0 & \text{if } x_1 \leq u \\
x_1 & \text{if } u \leq x_1 \leq l_0 \\
l_0 & \text{if } x_1 \geq l_0 
\end{cases}
\]
and \( B(x_1) = \sup \{ x_2 \geq 0 : \bar{y}^2(x_1, x_2|K_2) = 0 \} \). Then, all of the results in the theorems hold.

**Part II:** For the optimization problem \( K_1 \), it is easy to see that if \( x_1 \geq l_0 \) and \( x_2 \leq u \), then
\[
K_1(x_1, x_2) = (c-s)x_2 + J_t(0, u) + J_t(l_0, 0).
\]
Hence \( \bar{y}^2(x_1, x_2|K_1) = 0 \) for any \( x_1 \) and \( x_2 \).
For the optimization $K_2$, we consider the following cases.

Case 1. If $x_2 \leq 2u$, then $K_2(x_1, x_2) = (c - s)x_2 + J_t(0, u) + J_t(0, u)$. Since $\frac{x_2^2}{2} > (x_2|K_2)$ is decreasing in $x_2$, $\frac{x_2^2}{2} > (x_2|K_2) = 0$ for all $x_2$.

Case 2. If $2u \leq x_2 \leq 2u_0$, then $K_2(x_1, x_2) = (c - s)x_2 + J_t(0, x_2/2) + J_t(0, x_2/2)$.

From the above discussion, if we define $B(x_1) = C(x_1) = 0$, then all of the results in the theorem hold.

Part III: Since $J_t(l_0 - y, y)$ is convex in $y$, we let $\bar{l} = \sup\{y \in [0, l_0] : J_t(l_0 - y, y) \geq J_t(0, u)\}$. It is easy to see that $s - c + (\theta + \alpha u) \Phi(\bar{l}) \leq 0$. We first derive the optimal policy for the optimization problem $K_1$.

Case 1(i): If $x_1 \leq l_0$ and $x_2 \leq x_1$,

$$K_1 - (c - s)x_2 = \max_{y_1 \geq x_1} \{J_t(0, y_1^1) + J_t(x_1, 0)\}, \max_{y_1 \geq x_1} \{J_t(x_1, 0) \} = J_t(0, \max\{x_2, u\}) + J_t(x_1, 0).$$

In this case, if $x_1 \leq \bar{l}$, $K_1(x_1, x_2) = (c - s)x_2 + J_t(0, u) + J_t(x_1, 0)$.

Case 2(ii): If $x_1 \leq l_0$ and $x_2 > x_1$,

$$K_1 - (c - s)x_2 = \max_{y_1 \geq x_1} \{J_t(0, y_1^1) + J_t(0, 0)\}, \max_{y_1 \geq x_1} \{J_t(0, y_1^1) + J_t(0, 0)\} = J_t(0, \max\{x_2, u\}) + J_t(x_1, 0).$$

Case 2(iii): If $x_1 \geq l_0$ and $x_2 \leq 2l_0 - x_1$,

$$K_1 - (c - s)x_2 = \max_{y_1 \geq x_1} \{J_t(0, l_0) + J_t(l_0, 0)\}, \max_{y_1 \geq x_1} \{J_t(l_0, 0) \} = J_t(0, \max\{x_2, u\}) + J_t(x_1, 0).$$

Case 2(ii): If $x_1 \geq l_0$ and $2l_0 - x_1 \leq x_2 \leq l_0$,

$$K_1 - (c - s)x_2 = \max_{y_1 \geq x_1} \{J_t(l_0 - y_1^1, y_1^1) + J_t(l_0, 0)\}, \max_{y_1 \geq x_1} \{J_t(l_0, 0) \} = J_t(l_0, 0) + \max\{J_t(0, u), J_t(l_0 - x_2, x_2)\}.$$

Thus, if $x_2 \geq \bar{l}$, $K_1 = (c - s)x_2 + J_t(0, u) + J_t(0, 0)$. If $x_2 \leq \bar{l}$, $K_1 = (c - s)x_2 + J_t(l_0 - x_2, x_2) + J_t(l_0, 0)$.

Case 2(iii): If $x_1 \geq l_0$ and $x_2 \geq l_0$,

$$K_1 - (c - s)x_2 = \max_{y_1 \geq x_1} \{J_t(0, y_1^1) + J_t(l_0, 0)\}, \max_{y_1 \geq x_1} \{J_t(x_1 - l_0, y_1^1) + J_t(l_0, 0)\} = J_t(0, \max\{u, x_2\}) + J_t(l_0, 0).$$
Similarly, we can characterize the optimal policy for the optimization problem $K_2$. Notice that
\[
\frac{\partial J_t(x_1, y)}{\partial y} |_{y = l_0 - x_1} = (\theta + \alpha v)(\Phi(l_0 - x_1) - \frac{c - s}{\theta + \alpha v}).
\]
Let $\tilde{u} = \sup \{ x_1 : \Phi(l_0 - x_1) \geq \frac{c - s}{\theta + \alpha v} \}$. Thus, \( \max_{0 \leq y^2 \leq l_0 - x_1} J_t(x_1, y^2) = J_t(x_1, l_0 - x_1) \) for \( x_1 \leq \tilde{u} \) and \( \max_{0 \leq y^2 \leq l_0 - x_1} J_t(x_1, y^2) = J_t(x_1, O(x_1)) \) otherwise. Let \( u' = \inf \{ x_1 \geq \tilde{u} : J_t(x_1, O(x_1)) \geq J_t(0, u) \} \).

**Case 1(i):** If \( x_2 \leq u \) and \( x_1 \geq l_0 \),
\[
K_2 - (c - s)x_2 = \max_{0 \leq z_1^2 \leq x_1, y^2 \geq 0} \{ J_t(0, u) + J_t(z_1^2, y^2) \} = \max \{ \max_{l_0 - x_1 \leq y^2 \leq 0} \{ J_t(0, u) + J_t(0, y^2) \}, \max_{0 \leq y^2 \leq l_0 - x_1} J_t(0, u) + J_t(0, y^2) \} = J_t(0, u) + J_t(l_0, 0).
\]

**Case 1(ii):** If \( x_2 \leq u \) and \( x_1 \leq l_0 \),
\[
K_2 - (c - s)x_2 = \max_{0 \leq z_1^2 \leq x_1, y^2 \geq 0} \{ J_t(0, u) + J_t(z_1^2, y^2) \} = \max \{ \max_{l_0 - x_1 \leq y^2 \leq 0} \{ J_t(0, u) + J_t(l_0 - y^2, y^2) \}, \max_{0 \leq y^2 \leq l_0 - x_1} J_t(0, u) + J_t(x_1, y^2) \} = \max \{ J_t(0, u) + J_t(0, y^2) \}.
\]

In this case, if \( x_1 \leq u' \), \( K_2(x_1, x_2) = (c - s)x_2 + J_t(0, u) + J_t(0, u) \); and if \( x_1 \geq u' \), \( K_2(x_1, x_2) = (c - s)x_2 + J_t(x_1, 0) + J_t(x_1, O(x_1)) \).

**Case 2(i):** If \( x_1 \geq l_0 \) and \( x_2 - u \geq l_0 \),
\[
K_2 - (c - s)x_2 = \max \{ \max_{0 \leq z_1^2 \leq x_1, 0 \leq y^2 \leq x_2 - u} \{ J_t(0, x_2 - y^2) + J_t(z_1^2, y^2) \}, \max_{0 \leq z_1^2 \leq l_0, 0 \leq x_2 - u} \{ J_t(0, u) + J_t(z_1^2, y^2) \} \} = \max \{ \max_{0 \leq y^2 \leq l_0} \{ J_t(0, x_2 - y^2) + J_t(l_0 - y^2, y^2) \}, \max_{0 \leq y^2 \leq x_2 - u} \{ J_t(0, u) + J_t(l_0 - y^2, y^2) \} \}, \max_{0 \leq y^2 \leq x_2 - u} \{ J_t(0, u) + J_t(l_0, y^2) \} \}.
\]

**Case 2(ii):** If \( x_1 \geq l_0 \) and \( u \leq x_2 \leq u + l_0 \),
\[
K_2 - (c - s)x_2 = \max \{ \max_{0 \leq z_1^2 \leq x_1, 0 \leq y^2 \leq x_2 - u} \{ J_t(0, x_2 - y^2) + J_t(z_1^2, y^2) \}, \max_{0 \leq z_1^2 \leq l_0, 0 \leq x_2 - u} \{ J_t(0, u) + J_t(z_1^2, y^2) \} \} = \max \{ \max_{0 \leq y^2 \leq x_2 - u} \{ J_t(0, x_2 - y^2) + J_t(l_0 - y^2, y^2) \}, \max_{x_2 - u \leq y^2 \leq l_0} \{ J_t(0, u) + J_t(l_0 - y^2, y^2) \}, \max_{0 \leq y^2 \leq x_2 - u} \{ J_t(0, u) + J_t(l_0, y^2) \} \}.
\]

For case 2(i) and (ii), we can show that if \( x_2 \geq u + \tilde{l} \),
\[
K_2 - (c - s)x_2 = \max \{ \max_{0 \leq y^2 \leq \tilde{l}} \{ J_t(0, x_2 - y^2) + J_t(l_0 - y^2, y^2) \}, 2J_t(0, \max \{ u, x_2 / 2 \} ) \};
\]

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and if \( x_2 \leq u + \tilde{t} \),

\[
K_2 - (c - s)x_2 = \max_{0 \leq y^2 \leq x_2 - u} \{ J_t(0, x_2 - y^2) + J_t(l_0 - y^2, y^2) \}.
\]

Case 2(iii): If \( x_1 \leq l_0 \) and \( l_0 - x_1 \geq x_2 - u \geq 0 \),

\[
K_2 - (c - s)x_2 = \max\left\{ \max_{0 \leq y^2 \leq x_2 - u} \left\{ J_t(0, x_2 - y^2) + J_t(x_1, y^2) \right\}, \max_{l_0 - x_1 \leq y^2 \leq l_0} \left\{ J_t(0, u) + J_t(l_0 - y^2, y^2) \right\} \right\}.
\]

In this case, if \( x_1 \leq u' \), \( K_2(x_1, x_2) = (c - s)x_2 + J_t(0, u) + J_t(0, u) \). If \( x_1 \geq u' \) and \( x_2 - u \leq O(x_1) \),

\[
K_2(x_1, x_2) = (c - s)x_2 + J_t(0, u) + J_t(x_1, O(x_1)).
\]

And if \( x_1 \geq u' \) and \( x_2 - u \geq O(x_1) \),

\[
K_2 - (c - s)x_2 = \max\left\{ \max_{0 \leq y^2 \leq x_2 - u} \left\{ J_t(0, x_2 - y^2) + J_t(x_1, y^2) \right\}, J_t(0, u) + J_t(0, u) \right\}.
\]

Case 2(iv): If \( x_1 \leq l_0 \) and \( l_0 - x_1 \leq x_2 - u \leq l_0 \),

\[
K_2 - (c - s)x_2 = \max\left\{ \max_{l_0 - x_1 \leq y^2 \leq x_2 - u} \left\{ J_t(0, x_2 - y^2) + J_t(l_0 - y^2, y^2) \right\}, \max_{0 \leq y^2 \leq l_0 - x_1} \left\{ J_t(0, x_2 - y^2) + J_t(x_1, y^2) \right\} \right\}.
\]

Case 2(v): If \( x_1 \leq l_0 \) and \( x_2 - u \geq l_0 \),

\[
K_2 - (c - s)x_2 = \max\left\{ \max_{l_0 - x_1 \leq y^2 \leq x_2 - u} \left\{ J_t(0, x_2 - y^2) + J_t(l_0 - y^2, y^2) \right\}, \max_{0 \leq y^2 \leq l_0 - x_1} \left\{ J_t(0, x_2 - y^2) + J_t(x_1, y^2) \right\} \right\}.
\]

For Cases 2(iv) and (v), we can show that if \( x_1 \leq u' \), we have \( K_2 - (c - s)x_2 = 2J_t(0, \max\{x_2, u\}/2) \); and if \( x_1 \geq u' \), we have

\[
K_2 - (c - s)x_2 = \max\left\{ \max_{l_0 - x_1 \leq y^2 \leq l_0 - u} \left\{ J_t(0, x_2 - y^2) + J_t(l_0 - y^2, y^2) \right\}, \max_{O(x_1) \leq y^2 \leq l_0 - x_1} \left\{ J_t(0, x_2 - y^2) + J_t(x_1, y^2) \right\}, 2J_t(0, \max\{x_2, u\}/2) \right\}.
\]
Let \( C(x_1) = \inf\{x_2 \geq 0 : z^1_1(x_1, x_2|K_1) = 0\} \) and \( B(x_1) = \sup\{x_2 \geq 0 : y^2(x_1, x_2|K_2) = 0\} \), then all the results hold. In addition, from the above discussion, we know that if \( u' \leq l \), \( C(x_1) \) must be smaller than
\[
\tilde{C}(x_1) = \begin{cases} 
0 & \text{if } x_1 \leq l; \\
\tilde{l} & \text{if } x_1 \geq l
\end{cases}
\]
and \( B(x_1) \) must be greater than
\[
\tilde{B}(x_1) = \begin{cases} 
0 & \text{if } x_1 \leq l; \\
u & \text{if } x_1 \geq l.
\end{cases}
\]
If \( u' \geq l \), \( C(x_1) \) must be smaller than
\[
\tilde{C}(x_1) = \begin{cases} 
0 & \text{if } x_1 \leq u'; \\
\tilde{l} & \text{if } x_1 \geq u'
\end{cases}
\]
and \( B(x_1) \) must be greater than
\[
\tilde{B}(x_1) = \begin{cases} 
0 & \text{if } x_1 \leq u'; \\
u & \text{if } x_1 \geq u'.
\end{cases}
\]
\( \Box \)

**Proof of Theorem 9**: To show the result, it suffices to show that if \( z^1_1 + \bar{y}^1 \geq \bar{y}^2 \), then part (ii) must hold. The proof is achieved by contradiction. Suppose for all optimal policies that satisfy \( z^1_1 + \bar{y}^1 \geq \bar{y}^2 \), we have \( z^1_1 > 0 \) and \( \bar{y}^2 > 0 \).

Let \( \delta = \min(z^1_1, \bar{y}^2) \) and we construct a new policy as follows:
\[
\begin{align*}
z^1_1 &= \bar{z}^1_1 - \delta, \quad z^1_2 = \bar{z}^1_2 + \min(\delta, z^2_2), \quad y^1 = \bar{y}^1 + \delta, \\
z^2_1 &= \bar{z}^2_1 + \delta, \quad z^2_2 = \bar{z}^2_2 - \min(\delta, z^2_2), \quad y^2 = \bar{y}^2 - \delta,
\end{align*}
\]

It is not difficult to show that the new policy is still feasible and either \( z^1_1 \) or \( y^2 \) is zero and \( z^1_1, z^1_2, y^1, z^2_1, z^2_2, \) and \( y^2 \) are all nonnegative. The objective function \( J_i \) can be written as
\[
J_i(z^i_1, y^i) = -sz^i_1 - cy^i + pE\min(D^i, z^i_1 + y^i) - \theta E(z^i_1 + y^i - D^i)^+ + (\theta + \alpha v)E(y^i - D^i)^+.
\]

The expected profit under the new policy minus that under the optimal policy is
\[
(\theta + \alpha v)[E(\bar{y}^1 + \delta - D^1)^+ + E(\bar{y}^2 - \delta - D^2)^+ - E(\bar{y}^1 - D^1)^+ - E(\bar{y}^2 - D^2)^+].
\]

We have
\[
E(\bar{y}^1 + \delta - D^1)^+ - E(\bar{y}^1 - D^1)^+ \geq E(\bar{y}^2 - D^1)^+ - E(\bar{y}^2 - \delta - D^1)^+, \\
\geq E(\bar{y}^2 - D^2)^+ - E(\bar{y}^2 - \delta - D^2)^+.
\]
Here the first inequality holds because the function $E(x - D_1^1)$ is a convex function in $x$ and $\bar{y}^1 + \delta \geq \bar{y}^2$. The second is true because $(x - y)^+$ is submodular in $(x, y)$, and $D_2^2$ is larger than $D_1^1$ stochastically. □

References


