Separation of Perishable Inventories in Offline Retailing through Transshipment

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Abstract

Transshipment in retailing is a practice where one outlet ships its excess inventory to another outlet with inventory shortages. By balancing inventories, transshipment can reduce waste and increase fill rate at the same time. In this paper, we explore the idea of transshipping perishable goods with a fixed finite lifetime in offline grocery retailing. In the offline retailing of perishable goods, customers typically choose the newest items first, which can lead to substantial waste. We show that in this context, transshipment plays two roles. One is inventory balancing, which is well known in the literature. The other is inventory separation, which is new to the literature. That is, transshipment allows a retailer to put newer inventory in one outlet and older inventory in the other. This makes it easier to sell older inventory and reduces waste as a result. To understand how exactly inventories should be separated, we study an approximation model that relies on only two pieces of information, namely the number of items expiring in one period (old items) and that of the rest (new items). We show that the optimal policy can be characterized by two increasing switching curves. The two switching curves divide the entire state space into three regions. In the first region, only one outlet holds old items while both hold new items. In the second, one outlet holds old items and the other holds new items. In the third, only one outlet holds new items while both hold old items. We also conduct numerical studies to quantify the value of transshipment.

1 Introduction

The sale of perishable products accounts for over 50% of the business of grocery retailing (Laseter and Rabinovich 2012). As consumers become increasingly health conscious, the importance of perishable products can only grow. Besides, retailers also rely on perishable products to drive store traffic and gain competitive edge (Tsiros and Heilman 2005). Managing inventory of perishable products, however, is challenging and waste is substantial in retailing. Retailers remove items from the shelves when they are near or past their expiration dates. In America, approximately 40% of food produce is wasted, much of it in retailing (Kaye 2011). Waste in
retailing is a problem in many other places too. According to a recent study by Friends of the Earth, the four main supermarkets in Hong Kong throw away 87 tons of food a day, and most of the waste ends up in landfill (Wei 2012). Many retailers have programs aimed at addressing the problem. For example, the Food Waste Reduction Alliance in the United States, the Waste and Resources Action Programme in the United Kingdom, and the Retailers’ Environmental Action Programme in Europe were all established with waste reduction as their primary goal.

The main challenges in managing perishable products at offline grocery retailers are well known. First, demand is uncertain, and hence it is difficult to match supply with demand. Second, perishable items typically have very short lifetimes and hence they need to be sold in a short time window. Third, customers choose items on a last-in-first-out (LIFO) basis. Retailers replenish their shelves periodically. Whenever a new shipment arrives, the items on the shelves, which have shorter remaining lifetimes, are one step closer to the bin. Various ideas for managing perishable products have been adopted in practice and discussed in the academic literature. In particular, to induce more customers to choose older items first and hence leave fresher items on shelves, items on shelves can be arranged in a way that flesher items are harder to reach. But this practice can only make it harder for customers to retrieve the flesher items, not stop them from doing so. For the retailers with backroom storage, workers can stock shelves more frequently and in smaller quantity, effectively hiding fresher items from customers until older items are (almost) sold out. This tactic requires backroom refrigerated storage, constant monitoring of inventory levels on shelves and frequent replenishment. In this paper, we add a new weapon to the arsenal in the war against perishability: transshipment.

Transshipment has been widely studied in the operations management literature, but it has not been studied in retailing of perishable products under the LIFO rule. Existing research shows that its benefit comes from balancing inventories across different locations and hence reducing waste at some locations and shortages at others at the same time. In this study we explore the idea of transshipment in an offline retailer consisting of two outlets. The retailer replenishes its perishable products every period and at the end of each period, the retailer can either put the products that have not expired on clearance sale or carry them over to the next period. The products have a fixed finite lifetime. The retailer can also transship them from one outlet to the other. Our analysis shows that in this context, transshipment works very differently. It can balance inventories across different locations, similar to what is known in the literature. However, besides that, transshipment also plays a very different role in that it allows the separation of items with different remaining lifetimes. That is, transshipment allows
the retailer to put newer inventory in one outlet and older inventory in the other. This makes it easier to sell older inventory and reduces waste as a result.

To understand how exactly inventories should be separated, we consider an approximation that relies on only two pieces of information, namely the number of items expiring in one period (old items) and that of the rest (new items), we show that the optimal policy can be characterized by two increasing switching curves. The two switching curves divide the entire state space into three regions. In the first region, only one outlet holds old items while both hold new items. In the second, one outlet holds old items and the other holds new inventory. In the third, only one outlet holds new items while both hold old items. Our numerical studies show that transshipment and clearance sales are substitutes in terms of *both* increasing profit and reducing waste. Transshipment can increase profit by as much as several percentage points. It is most valuable in increasing profit when the variable cost of products is high, outdating cost is high, clearance sale price is low, or demand variability is high.

To turn the idea of transshipment of perishables into reality, three important issues must be considered. First, the value of perishables per unit is typically low. Therefore, transshipment is viable economically only if the scale is large enough and the logistics is extremely efficient. Ideally, the transshipment should be integrated with the existing replenishment process so that the additional variable cost is minimal. Second, transshipment between outlets may require the cooperation of store managers whose incentive may not be aligned with that of the retailer. Third, our analysis suggests that in each period the retailer put old inventory in one outlet and new inventory in the other. The two outlets should take turn to be the one which receives the old inventory so that in the long run, the two outlets have equally fresh inventories. These issues can be difficult, but not impossible, barriers to overcome. How to overcome these barriers is beyond the scope of our current research. Instead, we focus on how transshipment should be implemented and its impact on profit. We believe this is the first step and only then will we know whether it is worth overcoming these barriers in implementation. Practically, our results are directly useful for retailers of perishable goods. Theoretically, the study also provides a completely new perspective on transshipment, an important concept in the field of operations management, and enriches the literature on perishable inventory and that on transshipment.

The remainder of the paper is organized as follows. In Section 2, we review related literature. We present the general model and its properties in Section 3. The general problem is computationally challenging. Therefore, we discuss its approximation in 4. The effects of transshipment on profit and waste are tested numerically in Section 5. We discuss an extension
in Section 6 and conclude the paper in Section 7.

2 Related Literature

The study is directly related to two streams of literature in operations management. The first is the literature on transshipment. This literature is voluminous. The recent studies can be classified into two types. In the first type, there is a central decision maker who has access to full information and makes all the decisions. Representative studies of this type include, Abouee-Mehrizi et al. (2015) (lost sales), Hu et al. (2008) and Li and Yu (2014) (capacity constraints), and Yang and Zhao (2007) (virtual transshipment). In all these studies, the objective is to characterize and compute the optimal replenishment and transshipment policies. In the second type of studies, there are multiple decision makers with different incentives. Various research questions have been raised. For example, Hu et al. (2007) focused on the question of whether a pair of coordinating transshipment prices, i.e., payments that each party has to make to the other for the transshipped goods, can be set globally such that the local decision makers are induced to make inventory and transshipment decisions that are globally optimal. Dong and Rudi (2004) and subsequently Zhang (2005) studied how transshipments affect independent manufacturers and retailers in a supply chain where retailers can transship inventory. Studies also exist that consider the cooperation and competition of retailers using cooperative game theory (e.g., Sosic 2006, Fang and Cho 2014). None of these studies has considered perishable products with a general lifetime.

The second stream is the literature on perishable inventory. A considerable renewed interest exists in the area (see, for example, Chao et al. 2018, Chao et al. 2015, Chen et al. 2014, Li and Yu 2014, and reviews by Karaesmen et al. 2011 and Nahmias 2011). Particularly related to our study is the strand of literature that considers the LIFO rule. Cohen and Pekelman (1978) analyzed the evolution over time of the age distribution of inventory. Under two particular order policies, constant order quantity and fixed critical number, they determined the shortages and outdates in each period by the age distribution and related them to inventory decisions. Pierskalla and Roach (1972) and Deniz et al. (2010) considered issuing endogenously and the set of feasible issuing rules includes LIFO. The former showed that under most of the objectives, first-in-first-out (FIFO) is the optimal issuing rule. The latter focused on finding heuristics to coordinate replenishment and issuing. Parlar et al. (2008) and Cohen and Pekelman (1979) compared FIFO issuance with LIFO issuance. None of the above-mentioned papers has considered the optimal inventory ordering policy under LIFO.
In spite of the practical relevance of the LIFO rule to retailing, little work has been done, especially in terms of optimal policies, perhaps due to the technical difficulties. However, recent progress is encouraging. Li et al. (2016) focused on the optimal policies on inventory control and clearance sales under LIFO and a general life time. They showed that a clearance sale may occur if the level of inventory with a remaining lifetime of one period is either very high or very low, a phenomenon that is unique to the LIFO rule. Furthermore, they showed that myopic heuristics requiring only information about total inventory and information about the inventory with a remaining lifetime of one period performed consistently well. Li et al. (2017) examined the impact of shelf-life-extending packaging on the optimal policy, cost, and waste. One interesting insight they gave was that although it may not be optimal in terms of cost, the adoption of shelf-life-extending packaging can consistently reduce waste substantially. None has considered transshipment in the literature on perishable inventory with a general lifetime.

The study closest to ours is perhaps that of Zhang et al. (2017). They studied transshipment of perishable inventory with a general lifetime between two locations. However, they assumed a FIFO rule and exogenous order-up-to levels, neither of which holds in offline grocery retailing. In summary, we are the first to consider transshipment of perishable inventory in offline grocery retailing.

3 The General Formulation

There are two identical outlets, indexed by superscript \( i = 1, 2 \), owned by the same retailer. The products they sell have an \( n \)-period lifetime. The products can be sold either at a regular price, \( p \), or a clearance sale price, \( s \). Under a regular price, the demand at each outlet is random and is modeled by random variable \( D_i \). The demand under a clearance sale is so high (or \( s \) is so low) that inventory on clearance sales will never go unsold. More sophisticated pricing schemes have been used in services such as hotels and airlines, but are uncommon in offline retailing. We assume that \( D_1 \) and \( D_2 \) are identically but not necessarily independently distributed. The assumption is made so that we can sharpen the key insights and we will discuss the more general cases toward the end. Let \( \Phi(\cdot) \) and \( \phi(\cdot) \) denote the cumulative distribution function and the density function for the demand, respectively.

The timing of events is as follows. 1) At the beginning of a period, the retailer determines how much to order and how much and what should be transshipped from one outlet to the other. 2) Then the random demand for regular sales is realized. 3) At the end of the period, the unsold inventory with a remaining lifetime of one period expires; and 4) the retailer determines how
much of the inventory that has not expired should be carried over to the next period and how much should be put on clearance sale. Because there is no information updating between the ordering and transshipment decisions in 1) and the clearance sale decisions in 4), we redefine a period by moving 4) to the beginning of a period. In other words, all decisions are made at the beginning of a period. We assume that there is no transshipment cost in the model and the implication of transshipment cost will be discussed in the Conclusion section.

For outlet \( i \), the initial inventory is represented by a vector \( x^i = (x^i_1, x^i_2, \ldots, x^i_{n-1}) \), where \( x^i_j \) represents the inventory with a remaining lifetime of \( j \) periods at outlet \( i \). Let \( \mathbf{z}^i = (z^i_1, \ldots, z^i_{n-1}) \), where \( z^i_j \) is the inventory with a remaining life time of \( j \) periods that retail outlet \( i \) has after transshipment and clearance sale. As such, the total amount of inventory with a remaining lifetime of \( j \) periods available for regular sales is \( z^i_1 + z^i_2 \) and the amount sold in clearance sales is \( x_j - z^i_1 - z^i_2 \). Customers would always choose the freshest products first; that is, inventory leaves the retail shelf on a LIFO basis. Suppose the system state becomes \( Y^i(q^i, \mathbf{z}^i, D^i) = (Y^i_1, Y^i_2, \ldots, Y^i_{n-1}) \) in the next period. Then, for \( 1 \leq j \leq n - 2 \)

\[
Y^i_j(q^i, \mathbf{z}^i, D^i) = (z^i_{j+1} - (D^i + q^i - \sum_{k=j+2}^{n-1} z^i_k))^+ 
\]

and

\[
Y^i_{n-1}(q^i, \mathbf{z}^i, D^i) = (q^i - D^i)^+. 
\]

The outdated amount is

\[
S(q^i, \mathbf{z}^i, D^i) = (z^i_1 - (D^i + q^i - \sum_{j=2}^{n-1} z^i_j))^+. 
\]

Let \( c, \theta, \) and \( \alpha \) be the ordering cost, outdating cost, and the discounting factor, respectively. Without loss of generality, we assume that there is no holding cost. The dynamic programming formulation is as follows:

\[
J_t(\mathbf{z}^i, q^i) = -s \sum_{j=1}^{n-1} z^i_j - cq^i + pE \min(q^i + \sum_{j=1}^{n-1} z^i_j, D^i) - \theta ES(q^i, \mathbf{z}^i, D^i), 
\]  

(1)

and

\[
v_t(\mathbf{x}) = s \sum_{j=1}^{n-1} x_j + \max\{J_t(\mathbf{z}^1, q^1) + J_t(\mathbf{z}^2, q^2) + \alpha Ev_{t+1}(\sum_{i=1}^{2} Y^i(q^i, \mathbf{z}^i, D^i))\}, 
\]  

(2)
subject to $z_j^1 + z_j^2 \leq x_j$, $z_j^i \geq 0$, $q^i \geq 0$ for all $i = 1, 2$ and $j = 1, 2, \ldots, n-1$. On the right-hand side of (1), the second term is the purchasing cost, the third term the revenue from regular sales, and the last term the outdating cost. The sum of the first terms on the right-hand sides of (1) and (2) represents the revenue from clearance sales. Hence $J_t(z', q^i)$ is the one-period profit generated at outlet $i$. The planning horizon is $T$ and the terminal condition is $v_{T+1}(\cdot) = 0$. Denote by $(\bar{z}_j^i, \bar{q}_i)$, $j = 1, 2, \ldots, n-1$ and $i = 1, 2$, the optimal solution to (2). The optimal policy is time dependent. But to simplify notation, we omit the time index when there is no risk of confusion. We also assume that $s < c$ to avoid trivial solutions.

Let $e_i$ denote an $n-1$ dimensional unit vector where the $i$-th element equals one and all other elements equal zero. Let $\delta$ be a small positive number. We can show the following results on the marginal values of initial inventories.

Lemma 1

(i) $v_t(x + \delta e_i) \leq v_t(x + \delta e_{i+1})$;

(ii) $s\delta \leq v_t(x + \delta e_i) - v_t(x) \leq c\delta$;

(iii) $J_t(z', q^i)$ is submodular in $(z_1^i, q^i)$ and $(z_1^i, z_j^i)$ for $j \geq 2$.

In the next theorem, we show that if items with a two-period or longer lifetime are sold through clearance sales under the optimal policy, then all the older inventories are cleared and no new items are ordered. In addition, if the total inventory on hand with a remaining lifetime of at least two periods is large enough, then at least one of the two outlets will not be keeping inventory with a remaining lifetime of one period. Let $l_0 = \Phi^{-1}(\frac{s+\theta}{p+\theta})$, which represents the optimal quantity of inventory with a remaining lifetime of one period an outlet should carry over to the next period when the selling price is $p$, disposal cost is $\theta$, and the opportunity cost (clearance price) is $s$.

Theorem 1

(i) If $\bar{z}_i^1 + \bar{z}_i^2 < x_i$ for some $i \geq 2$, then $\bar{z}_j^1 = \bar{z}_j^2 = 0$ for all $j < i$ and $\bar{q}_1^1 = \bar{q}_i^2 = 0$;

(ii) If $\sum_{i=2}^{n-1} x_i \geq 2l_0$, then either $\bar{z}_1^i = 0$ or $\bar{z}_j^1 = 0$.

In Part (ii) of Theorem 1, if the optimal solution is symmetrical, then both $\bar{z}_1^i$ and $\bar{z}_j^1$ should be zero. However, the optimal solution may not be symmetrical, even though the two outlets are identical and face identically distributed demands. Indeed, when the inventory is depleted
on an FIFO basis, we can show that there is a symmetrical optimal solution. However, this is not the case in our setting.

The following result requires the random demands to follow a $PF_2$ distribution. $PF_2$ distributions are also known to have log-concave densities (Ross 1983). This is a common assumption in the inventory literature (e.g., Huggins and Olsen 2010, Li et al. 2016), and the class of distributions includes many commonly used distributions. $PF_2$ distributions have the following smoothing property: if $D$ is a $PF_2$ random variable and $f(x)$ is quasiconcave, then $E f(x - D)$ is quasiconcave.

**Theorem 2** Suppose that $D^i$ has a $PF_2$ distribution. If $\bar{z}_1^1 = \bar{z}_1^2$ for $2 \leq i \leq n - 1$, then there is an optimal policy such that at least one of $\bar{z}_1^1$, $\bar{z}_1^2$, $\bar{q}_1$ and $\bar{q}_2$ is zero.

In the above theorem and the following Theorems 4 and 9, we show that there is an optimal policy that possesses certain properties. If the cumulative distribution functions of the demands are strictly increasing, then we can show that all optimal polices possess the properties. Theorem 2 includes two special cases. The first case is when $x_i = 0$ for all $i = 2, ..., n - 1$, and the second is when the lifetime $n = 2$. In both cases, the condition $\bar{z}_1^1 = \bar{z}_1^2$ for $2 \leq i \leq n - 1$ is obviously satisfied. Transshipment allows the retailer to send the oldest inventory to one outlet and the newest inventory to the other (i.e., separation of inventories) as well as send inventory from the outlet with excess inventory to the one with shortage (i.e., balance of inventories).

When exactly one of $\bar{z}_1^1$, $\bar{z}_1^2$, $\bar{q}_1$ and $\bar{q}_2$ is zero, or when exactly two of them are zero and one outlet holds only the oldest inventory and the other holds only the newest inventory, separation of inventories occurs. The following theorem looks at the similar issues from a different angle.

**Theorem 3** If $\bar{z}_1^1 > 0$ and $\bar{z}_1^2 > 0$, then $\sum_{j=1}^{n-1} \bar{z}_j^1 + \bar{q}_1 = \sum_{j=1}^{n-1} \bar{z}_j^2 + \bar{q}_2$.

According to Theorem 3, if both outlets hold the oldest inventory, then inventory is balanced in the sense that the two outlets hold the same amount of total inventory. In light of Part (ii) of Theorem 1, both outlets hold the oldest inventory when the total inventory with a remaining lifetime of two periods or longer is not too large. However, although the two outlets hold the same amount of inventory, the types of inventory they hold may be different and each may hold a certain type of inventory that the other does not hold. In these situations, separation of inventories occur.
4 Approximations

The structural properties in Section 3 provide useful guidance. However, to put the ideas into practice, there are still open questions. First, how much should each outlet order in each period, and how much existing inventories should be sold in clearance sales and how much should be carried over to the next period? Second, how should the inventories be allocated between the two outlets? Under what conditions should inventories be separated and what conditions they be balanced? Third, what would be the impact of transshipment on profit and waste? To answer these questions with the general formulation in Section 3, we need to know how many units of inventory there are in each age group, and with that information, to solve a dynamic program with a multi-dimensional state space and a non-concave objective function. The former is impossible given the current bar code design and standard and the latter is challenging computationally. Approximation is the only way forward.

Based on the ideas from Li et al. (2016), we simplify the general formulation in two steps. First, we approximate the profit-to-go by a linear function. That is, we let $v_t(x) = v \sum_{j=1}^{n-1} x_j$ for $t = 1, 2, ..., T$, where $v$ is a number bounded by $c$ and $s$ (e.g., $v = (s + c)/2$) because the marginal value of inventory is bounded by $c$ and $s$. Second, we aggregate the state variables $(x_2, x_3, ..., x_{n-1})$; that is, we look for policies that rely only on $x_1$ and the sum of $x_2, x_3, ..., x_{n-1}$. Retailers typically check and remove expired items manually. The process of putting items on clearance sales is also manual. The information about $x_1$ can be obtained during these manual processes.

The above approximation is appealing for the following reasons. First, the optimal solution in the approximation is myopic and easy to compute. Second, the implementation of optimal solution requires only two pieces of information - the oldest inventory and the total inventory. Third, the properties related to the original problem in Section 3 continue to hold true in the approximation model. Specifically, the marginal value of inventory is bounded by $c$ and $s$, newer inventory is more valuable than older inventory (Lemma 1), older inventory should be cleared before newer inventory, and ordering occurs only when inventory with a remaining lifetime equal to or longer than two periods is not sold in clearance sales (Theorem 1). In the original model we have shown that the retailer may separate the oldest inventories from the newest ones through transshipment. In the approximation, the incentive to separate inventories remain and we know exactly how the separation should be implemented. Finally and most importantly, it allows us to fully characterize the optimal solution analytically. Not only does it give us a good approximation that is easily computable, it also enhances our understanding of separation of
inventories, a phenomenon new to the field.

In this section, we continue to use \( x_1 \) to represent the inventory with a remaining lifetime of one period, but use \( x_2 \) to represent the total inventory with a remaining lifetime of two periods or longer. For ease of exposition, we call the former *old inventory* and the latter *new inventory*. We use \( z_1^1 \) and \( z_2^1 \) to represent the amount of old inventory and new inventory, respectively, at outlet \( i \) on regular sale. Let \( y_i^1 \) be the amount of new inventory after ordering at outlet \( i \). That is, \( y_i^1 \) is the order-up-to level for new inventory at outlet \( i \). To avoid the need for additional notation, we continue to use \( J_t \) and \( v_t \) to represent respectively the one-period profit for an outlet and the total maximal profit when the above approximations are used. Essentially, we are solving the following optimization program:

\[
J_t(z_1^1, y_i^1) = -sz_i^1 - cy_i^1 + pE \min(D_i^i, z_1^1 + y_i^1) - \theta E(z_1^1 - (D_i^i - y_i^1)^+) + \alpha v(y_i^1 - D_i^i),
\]

and

\[
v_t(x_1, x_2) = s(x_1 + x_2) + \max \{(c - s)(z_1^2 + z_2^2) + J_t(z_1^1, y_i^1) + J_t(z_2^1, y_i^2)\}
\]

subject to \( z_1^j + z_2^j \leq x_j, \ z_1^j \geq 0, \ y_i^1 \geq z_2^j \) for all \( i = 1, 2 \) and \( j = 1, 2 \). Denote by \( (\bar{z}_1^1, \bar{z}_2^1, \bar{y}_1^1) \) the optimal solution to (3).

The first part of the following lemma is similar to Lemma 1 (iii). The second part shows that after we approximate the complex value function by a linear function, the objective function is quasiconcave and hence it is easier to compute the optimal solution.

**Lemma 2**

(i) \( J_t(z_1^1, y_i^1) \) is submodular in \( (z_1^1, y_i^1) \);

(ii) If \( D_i^i \) has a PF_2 distribution, then \( J_t(z_1^1, y_i^1) \) is quasiconcave in \( y_i^1 \).

The following theorem is a stronger version of Theorems 2 and 3.

**Theorem 4** There is an optimal policy of (3) such that at least one of the two outlets holds either the old inventory or the new inventory, but not both types of inventory; that is, at least one of \( \bar{z}_1^1, \bar{z}_2^1, \bar{y}_1^1 \) and \( \bar{y}_2^2 \) is zero. In particular, there are three possibilities:

(i) \( \bar{z}_1^1 = \bar{z}_2^1 = 0 \). In this case, \( \bar{y}_1^1 = \bar{y}_2^2 \);

(ii) \( \bar{z}_1^1 > 0, \bar{z}_2^1 > 0 \). In this case, \( \bar{z}_1^1 + \bar{y}_1^1 = \bar{z}_2^2 + \bar{y}_2^2 \) and at least one of \( \bar{y}_1^1 \) and \( \bar{y}_2^2 \) is zero.
(iii) Either $z_1^1 > 0$ or $z_1^2 > 0$ but not both.

As we mentioned earlier, the retailer may use transshipment to separate or balance inventories, depending on the context. In case (i) above, the solution is symmetric and the inventories are balanced. In case (ii), both separation and balance of inventories occur. All the new inventory is sent to one outlet, but the two outlets hold the same total inventory. In case (iii), all the old inventory is sent to one outlet and inventories are separated. In the following several theorems, we will provide specific conditions, in terms of the state variables, under which each case will occur. Let $u = \arg \max_{y \geq 0} J_t(0, y)$. Then, $u = \Phi^{-1}(\frac{p-c}{p-\alpha v})$ and it represents the optimal ordering quantity of new items when the initial inventory of old items is zero. Let $u_0 = \arg \max_{y \geq 0} \{ (c-s)y + J_t(0, y) \}$. Then $u_0 = \sup \{ x : \Phi(x) \leq \frac{p-s}{p-\alpha v} \}$.

**Theorem 5**

(i) If $x_2 \geq 2u_0$, then it is optimal to clear all of the old inventory, clear the new inventory down to $2u_0$ and allocate it equally between the two outlets, and order nothing; that is, $\bar{y}_1 = \bar{y}_2 = u_0$, and $\bar{z}_1^1 = \bar{z}_2^1 = 0$.

(ii) There exists an increasing function $A(x_2)$ such that if $x_1 \leq A(x_2)$, then all $x_1$ is cleared.

Part (i) of Theorem 5 is related to Theorem 1. But the approximation allows for a stronger result. Based on Theorem 5 (ii) and Theorem 4, if $x_1$ is small or $x_2$ is large, then the two outlets hold exactly the same level of inventory after transshipment (i.e., balance of inventories). Otherwise, there is an asymmetric optimal solution (i.e., separation of inventories). Because of Theorems 1 and 4, the optimization problem (3) is equivalent to the following:

$$v_t(x_1, x_2) - s(x_1 + x_2) = \max \{ K_1(x_1, x_2), K_2(x_1, x_2), K_3(x_1, x_2) \}$$

where

$$K_1(x_1, x_2) = \max_{z_1^1 + z_1^2 \leq x_1, y_1 \geq x_2, z_1^i \geq 0} \{ (c-s)x_2 + J_t(z_1^1, y_1) + J_t(z_1^2, 0) \},$$

$$K_2(x_1, x_2) = \max_{0 \leq z_2^1 \leq x_1, y_1 + y^2 \geq x_2, y^i \geq 0} \{ (c-s)x_2 + J_t(0, y_1) + J_t(z_2^1, y^2) \},$$

and

$$K_3(x_1, x_2) = \max_{z_2^1 + z_2^2 \leq x_2, z_2^i \geq 0} \{ (c-s)(z_2^1 + z_2^2) + J_t(0, z_2^1) + J_t(0, z_2^2) \}.$$

Here $K_1$ represents the case when there are no clearance sales of new items, and all new items are allocated to outlet 1. $K_2$ represents the case when there are no clearance sales of new
items, and all old items are allocated to outlet 2. $K_3$ represents the case when some or all new items are sold in clearance sales, in which case, no order is placed and all old items are sold in clearance sales. In light of Theorems 1 and 4, these events are collectively exhaustive.

The optimal separation policy is given in the next theorem.

**Theorem 6** There exist two increasing functions $B(x_1) \geq C(x_1)$ such that

(i) If $x_2 > B(x_1)$, then at most one outlet holds old inventory while both outlets hold new inventory;

(ii) If $x_2 < C(x_1)$, then at most one outlet holds new inventory while both outlets hold old inventory;

(iii) If $C(x_1) \leq x_2 \leq B(x_1)$, then one outlet holds old inventory while the other holds new inventory.

Theorem 6 provides an optimal mapping between the allocation of inventories and the state. It reconfirms that it is suboptimal to stock both old and new inventories in both outlets. In addition, in light of Theorem 4, when $x_2 < C(x_1)$, the two outlets hold the same amount of total inventory. We further characterize the structure of the optimal clearance sales policy of old items in the following theorem.

**Theorem 7**

(i) For $x_2 < C(x_1)$, the old inventory is put on clearance sales if and only if $x_2 \geq 2l_0 - x_1$.

(ii) For $C(x_1) \leq x_2 < B(x_1)$, the old inventory is put on clearance sales if and only if $x_1 \geq l_0$.

(iii) For $x_2 > B(x_1)$, there exists a function $e_0(x_1)$ such that the old inventory is put on clearance sales if and only if $x_2 \geq e_0(x_1)$.

The results in Theorems 6 and 7 are shown in Figure 1. The clearance and non-clearance regions are labeled by “C” and “NC”, respectively. In Figure 1, in the region between $C(x_1)$ and $B(x_1)$, one outlet holds old inventory and the other holds new inventory; that is, old and new inventories are separated. In this case, if $x_1$ is greater than $l_0$, then $x_1$ is cleared down to $l_0$; otherwise, there is no clearance sale of $x_1$. This is consistent with Theorem 4 where we show that inventories are separated if and only if there is old inventory in the system after clearance sales. In general, old items are more likely to be put on clearance sale as $x_2$ increases. We can also see from the figure that for a given $x_2$, when $x_1$ is small enough, all $x_1$ should be cleared. The structure of the optimal ordering policy is given in the following theorem.
Figure 1: Separation of Inventories and Clearance Sales (Note: Clearance and Non-clearance Regions are Labeled by “C” and “NC”)

**Theorem 8**

(i) If $x_2 < C(x_1)$, there exists a function $e_1(x_2) \leq 2l_0 - x_1$ such that new items are ordered if and only if $x_1 \leq e_1(x_2)$.

(ii) If $C(x_1) \leq x_2 < B(x_1)$, new items are ordered if and only if $x_2 \leq u$.

(iii) If $x_2 > B(x_1)$, there exists a decreasing function $e_2(x_1)$ such that new items are ordered if and only if $x_2 \leq e_2(x_1)$.

The results in Theorem 8 can be visualized in Figure 2. The ordering and non-ordering regions are labeled by “O” and “NO”, respectively. The optimal ordering quantity is monotonically decreasing if the inventory level of old items increases, but not necessarily decreasing when there are more new items. Similar results have been established by Li et al. (2016). Technically, this happens because even though $J_t(z_i^1, y^i)$ is quasiconcave in $y^i$, $\max_{z_i \geq 0} J_t(z_i^1, y^i)$ is not necessarily quasiconcave in $y^i$. 

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5 Numerical Studies

In this section, through numerical studies, we first test the performance of the linear approximation. We then investigate two related issues. First, clearance sales and transshipment are two tools for retailers to fight perishability, but how are they related in creating value in terms of increasing profit and reducing waste? Second, under what circumstances does transshipment create the most value in increasing profit? For all the numerical studies, when the lifetime is no greater than 3, we compute the optimal policy for the dynamic programming formulation in Section 3. For any longer lifetimes, we compute the optimal policy for the approximated formulation in Section 4.

5.1 The accuracy of the approximation

In this subsection, we investigate the accuracy of the linear approximation that we proposed in Section 4. We also compare its performance with that of a quadratic approximation of the profit-to-go function, i.e., \( v_t(x) = \nu_1(\sum_{j=1}^{n-1} x_j)^2 + \nu_0(\sum_{j=1}^{n-1} x_j) \). We set lifetime \( n = 3 \). The approximation error is given by \( \text{Error} = \frac{v_1(0,0) - v_1^4(0,0)}{v_1(0,0)} \cdot 100\% \), where \( v_1^4(0,0) \) is the value function at the beginning of the planning horizon with initial states \((0,0)\) under one of the
two approximations. To calculate $v_1^A(0,0)$, we search for the optimal $v$, $\nu_0$ and $\nu_1$ for each approximation that lead to the maximum value of $v_1^A(0,0)$.

The numerical results are summarized in Table 1. The linear approximation is surprisingly good and is close to optimal when the margin of regular sales is not too low. The good performance of the linear approximation is probably because the value function in the original dynamic program is bounded by $s$ and $c$, and in spite of its simplicity, the linear approximation has already utilized the most crucial information - the oldest inventory and the total inventory. The quadratic approximation is better than the linear approximation, but only slightly.

<table>
<thead>
<tr>
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<tbody>
<tr>
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<tr>
<td>Linear</td>
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</tr>
<tr>
<td></td>
<td>$\theta = 4$</td>
<td>0.45</td>
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<tr>
<td></td>
<td>$\theta = 6$</td>
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</tr>
<tr>
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<td>$\theta = 0$</td>
<td>0.07</td>
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<tr>
<td></td>
<td>$\theta = 2$</td>
<td>0.40</td>
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<td></td>
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<td>$\theta = 2$</td>
<td>0.21</td>
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<td>0.45</td>
</tr>
<tr>
<td></td>
<td>$\theta = 6$</td>
<td>0.62</td>
</tr>
</tbody>
</table>

Note. $T = 10, n = 3, r = 10, \alpha = 0.99, D = Normal(3, \sigma)$ truncated below by 0.

### 5.2 Value of transshipment and clearance sales

In this subsection, we study the value of transshipment and clearance sales in terms of both improving profit and reducing waste. We let the lifetime be 3 and calculate the profit and waste under the optimal policy of the dynamic program at initial state $(0,0)$ for four different scenarios depending on whether or not transshipment and/or clearance sales are adopted. The
results are presented in Table 2.

Table 2: Value of transshipment and clearance sales.

<table>
<thead>
<tr>
<th>c = 3, θ = 1, s = 1</th>
<th>Transshipment</th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Profit</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Clearance</td>
<td>Y</td>
<td>174.76</td>
<td>174.16</td>
<td>0.60</td>
</tr>
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<td></td>
<td>N</td>
<td>174.08</td>
<td>173.16</td>
<td>0.92</td>
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<tr>
<td></td>
<td>Diff</td>
<td>0.68</td>
<td>1.00</td>
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</tr>
<tr>
<td>Waste</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Clearance</td>
<td>Y</td>
<td>2.85</td>
<td>2.49</td>
<td>0.36</td>
</tr>
<tr>
<td></td>
<td>N</td>
<td>4.12</td>
<td>4.37</td>
<td>-0.25</td>
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<tr>
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<td>Diff</td>
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<td>-1.88</td>
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</table>

<table>
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<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Profit</td>
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<td></td>
<td></td>
</tr>
<tr>
<td>Clearance</td>
<td>Y</td>
<td>53.45</td>
<td>51.49</td>
<td>1.96</td>
</tr>
<tr>
<td></td>
<td>N</td>
<td>53.45</td>
<td>51.49</td>
<td>1.96</td>
</tr>
<tr>
<td></td>
<td>Diff</td>
<td>0.00</td>
<td>0.00</td>
<td></td>
</tr>
<tr>
<td>Waste</td>
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<td>-0.63</td>
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<td>Diff</td>
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</thead>
<tbody>
<tr>
<td>Profit</td>
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<td></td>
<td></td>
</tr>
<tr>
<td>Clearance</td>
<td>Y</td>
<td>173.02</td>
<td>172.55</td>
<td>0.47</td>
</tr>
<tr>
<td></td>
<td>N</td>
<td>169.95</td>
<td>168.78</td>
<td>1.17</td>
</tr>
<tr>
<td></td>
<td>Diff</td>
<td>3.07</td>
<td>3.77</td>
<td></td>
</tr>
<tr>
<td>Waste</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Clearance</td>
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<td>0.27</td>
<td>0.30</td>
<td>-0.03</td>
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<td>Diff</td>
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</tr>
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<tbody>
<tr>
<td>Profit</td>
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<td></td>
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</tr>
<tr>
<td>Clearance</td>
<td>Y</td>
<td>52.85</td>
<td>52.23</td>
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<td>51.74</td>
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<td>Waste</td>
<td></td>
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<td></td>
</tr>
<tr>
<td>Clearance</td>
<td>Y</td>
<td>1.18</td>
<td>1.23</td>
<td>-0.05</td>
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<td>N</td>
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<td>-0.56</td>
</tr>
<tr>
<td></td>
<td>Diff</td>
<td>-0.51</td>
<td>-1.02</td>
<td></td>
</tr>
</tbody>
</table>

**Note.** T = 15, n = 3, r = 10, α = 0.99, D = Uniform[0, 2].

There are two observations from the table. First, clearance sales and transshipment are substitutes in both improving profit and reducing waste; i.e. having clearance sales decreases the value of having transshipment in improving profit or reducing waste. Second, in the presence of clearance sales, transshipment may not always reduce waste. This happens for example when c = 3, θ = 1, and s = 1. Since transshipment may reduce the need for clearance sales, the outlets may carry more old inventory to future periods with transshipment. In addition, the retailer may order more when transshipment is used. Both may increase waste.
5.3 The impact of transshipment on profit

For $n \leq 3$, we can calculate the difference in profit between the optimal policy of the dynamic program with transshipment and that without. For $n \geq 4$, which might be more realistic, computing the optimal policies of the dynamic program is no longer possible. Instead, we compare the performance of the heuristic policy with transshipment against that of the same heuristic policy but without transshipment (i.e., the two outlets are managed independently using the same myopic policies). We consider a 15-period horizon, i.e. $T = 15$. The values taken by system parameters are: $s \in \{0, 1, 2, 3\}$ when $c = 4$, $s \in \{0, 2, 4, 6\}$ when $c = 8$, and $\theta \in \{0, 2, 4, 6\}$. The demand distribution is assumed to be normal with mean $\mu = 3$, standard deviation $\sigma = 1$, and truncated below by zero. The discount rate $\alpha = 0.99$, and the regular sales price $r$ is held constant at 10.

When $n \geq 4$, an important issue when using the approximation is that the approximation only tells us the amount of total new inventory $z_j^i$ at each location $i$. However, there are many ways to allocate inventory across the two outlets such that $\sum_{j=2}^{n-1} z_j^i = z_j^2$ for $i \in \{1, 2\}$ and $\sum_{i=1}^{2} z_j^i \leq x_j$. The way inventory is allocated affects the initial inventory in the next period and in turn the profit and waste.

We use the sequential method in Algorithm 1 to allocate inventory to maximize the “freshness” of the initial inventory in the next period. The main idea is as follows. In outlet $i$, the newest initial inventory in the next period is $(\bar{q}^i - D^i)^+$, which is independent of the allocation. The second newest inventory is $(z_{n-1}^i - (D^i - \bar{q}^i)^+)^+$, which is supermodular in $(z_{n-1}^i, \bar{q}^i)$. Therefore, if $\tilde{q}^i \geq q_{n-1}^i$, we should allocate as much as needed there and allocate the remaining $x_{n-1}$ to outlet $i$ as needed there and allocate the remaining $x_{n-1}$ to outlet $i$. In other words, we allocate inventory from newest to oldest and the outlet with a greater $\tilde{q}^i$ has a higher priority in receiving allocation.

We use two examples to illustrate the algorithm. In both examples, we assume $n = 6$ and $(x_1, x_2, \ldots, x_5) = (5, 1, 1, 1, 2)$. In the first example, suppose that the optimal solution for the approximation model is $(\bar{z}_1^1, \bar{z}_2^1, \bar{y}^1) = (0, 2, 3)$ and $(\bar{z}_1^2, \bar{z}_2^2, \bar{y}^2) = (1, 3, 3)$. In this case, $(\tilde{q}^1, \tilde{q}^2) = (1, 0)$ and hence outlet 1 has a higher priority. The inventory composition after allocation at outlets 1 and 2 are $(0, 0, 0, 0, 2)$ and $(1, 1, 1, 1, 0)$, respectively. In the second example, suppose that the optimal solution for the approximation model is $(\bar{z}_1^1, \bar{z}_2^1, \bar{y}^1) = (2, 2, 2)$ and $(\bar{z}_1^2, \bar{z}_2^2, \bar{y}^2) = (0, 3, 4)$. In this case, $(\tilde{q}^1, \tilde{q}^2) = (0, 1)$ and hence outlet 2 has a higher priority. The inventory composition after allocation at outlets 1 and 2 are $(2, 0, 1, 1, 0)$ and $(0, 0, 0, 1, 2)$, respectively.
Algorithm 1 Allocation of new inventory across the two outlets

for $j = n - 1$ to 2 do
  · find $i$ such that $\bar{q}^i \geq \bar{q}^{3-i}$
  · $z_j^i = \min\{x_j, \bar{z}_2^i\}$
  · $x_j \rightarrow x_j - z_j^i$
  · $\bar{z}_2^i \rightarrow \bar{z}_2^i - z_j^i$
  · $z_j^{3-i} = \min\{x_j, \bar{z}_2^{3-i}\}$
  · $\bar{z}_2^{3-i} = \bar{z}_2^{3-i} - z_j^{3-i}$
  · $\bar{q}^i \rightarrow \bar{q}^i + z_j^i$
  · $\bar{q}^{3-i} \rightarrow \bar{q}^{3-i} + z_j^{3-i}$
end for

We summarize the numerical results in Table 3. In these studies, the increase in profit as a result of transshipment can be more than 4%, which is nontrivial given that grocery retailing is a low-margin business. Transshipment can increase profit the most when the ordering cost $c$ is high, clearance sale price $s$ is low, the outdating cost $\theta$ is high, and demand variability is high. When $c$ is high or $s$ is low, using clearance sale strategy to clear inventories approaching their expiration dates is costly. As a result, transshipment can have a greater impact. When $\theta$ increases, the outdating cost eats into the retailer’s profit in a more significant way. Transshipment, which can reduce waste if other things being equal, again has a greater impact. Finally, the effect of variability is consistent with what we know about pooling, of which transshipment is an example.

6 Non-identically Distributed Demands

In the earlier analysis, to avoid the obfuscation of main insights, we have assumed that the two outlets face identically distributed demands. Now let us consider a more realistic case where one outlet faces a stochastically larger demand than the other. Without loss of generality, suppose $D^2$ is stochastically larger than $D^1$ (Ross 1983); that is, $D^2 \geq_{st} D^1$. According to our analysis, it is not a good idea to put old and new inventory in the same outlet. When the demands are not identically distributed, a new issue arises: which outlet should get the new inventory and which outlet should have the old inventory? On the one hand, it makes sense to allocate the old inventory to outlet 2 because its stochastically larger demand makes it more likely to deplete the old inventory. On the other hand, outlet 2 requires a higher total inventory to fill its larger demand. If outlet 2 stocks a large amount of new inventory, which would make it difficult to
Table 3: Percentage Increase in Profit as a Result of Transshipment.

<table>
<thead>
<tr>
<th></th>
<th>( n = 2 )</th>
<th>( n = 3 )</th>
<th>( n = 4 )</th>
<th>( n = 5 )</th>
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<tr>
<td></td>
<td>( \sigma = 1 )</td>
<td>( \sigma = 2 )</td>
<td>( \sigma = 2 )</td>
<td>( \sigma = 2 )</td>
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<tr>
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<td>( c = 4 )</td>
<td>( c = 8 )</td>
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<td>( c = 8 )</td>
</tr>
<tr>
<td>( \theta = 0 )</td>
<td>0.64</td>
<td>0.97</td>
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<td>( \theta = 2 )</td>
<td>0.78</td>
<td>1.13</td>
<td>0.51</td>
<td>0.28</td>
</tr>
<tr>
<td>( \theta = 4 )</td>
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<td>1.12</td>
<td>0.60</td>
<td>0.31</td>
</tr>
<tr>
<td>( \theta = 6 )</td>
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<td>1.36</td>
<td>0.75</td>
<td>0.47</td>
</tr>
<tr>
<td>( s = 0 )</td>
<td>0.63</td>
<td>0.98</td>
<td>0.35</td>
<td>0.35</td>
</tr>
<tr>
<td>( s = 1 )</td>
<td>0.65</td>
<td>0.79</td>
<td>0.37</td>
<td>0.23</td>
</tr>
<tr>
<td>( s = 2 )</td>
<td>0.24</td>
<td>0.38</td>
<td>0.18</td>
<td>0.12</td>
</tr>
<tr>
<td>( s = 3 )</td>
<td>0.93</td>
<td>1.31</td>
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<td>0.35</td>
</tr>
<tr>
<td>( s = 4 )</td>
<td>0.91</td>
<td>1.32</td>
<td>0.65</td>
<td>0.65</td>
</tr>
<tr>
<td>( s = 6 )</td>
<td>0.83</td>
<td>1.35</td>
<td>0.70</td>
<td>0.70</td>
</tr>
</tbody>
</table>

Note. \( T = 15, r = 10, \alpha = 0.99, D = \text{Normal}(3, \sigma) \) truncated below by 0.
sell old inventory, then the old inventory should be rotated to outlet 1. The following theorem is a generalization of Theorem 4.

**Theorem 9** Suppose that $D^2 \geq_{st} D^1$. There is an optimal policy of (3) that satisfies at least one of the following statements:

(i) $\bar{z}_1 + \bar{y} \leq \bar{y}^2$;

(ii) $\bar{z}_2 > 0$ and either $\bar{z}_1$ or $\bar{y}^2$ is zero.

The above theorem offers specific guidelines for the retailer to follow. When there is a lot of old inventory, as much of it as needed at outlet 2 should first be sent there. It is easier to sell the old inventory at outlet 2 because of the greater demand and lower level of new inventory there. In this case, outlet 1 also holds old inventory if and only if the total old inventory exceeds what is needed at outlet 2 and hence $\bar{y}^2 = 0$ (Part (ii)). When there is only a small amount of old inventory, however, to meet the greater demand at outlet 2, a large amount of new inventory is needed there, which makes it hard to sell the old inventory. In this case, the old inventory should be first sent to outlet 1 (Part (i)). The total inventory at outlet 1 is low not only because the demand is lower there, but also because the level of new inventory there is intentionally kept low to reduce waste.

7 Conclusions

In this study, we explore the idea of transshipment in the context of retailing of perishable goods. Whether or not transshipment is worth implementing depends on its benefits and costs. The practice can increase profit by as much as several percentage points and we identify conditions under which it has the most benefit. Without a rigorous analysis like ours, such level of clarity is impossible. And as long as transshipment can be efficiently integrated into the regular replenishment process, the additional variable costs should be small. Implementation of transshipment may require changes in the business process, IT systems, and employee training upfront. These are all fixed costs, which do not affect our analysis and conclusions. Any retailers interested in the idea can weigh the benefits against the costs upfront and make an informed decision.

When there is a linear transshipment cost, we would expect that the lower the transshipment cost, the more likely separation of inventories may occur. This is confirmed in the numerical studies reported in Table 4. (Here $t$ denotes the unit transshipment cost.) We can also see from
the table that when the outdating cost is high or when the ordering cost is high, separation of inventories also occurs. This is expected because under these conditions the need for transshipment to reduce waste is high. Moreover, when the clearance sale price is high, the retailer may clear all the oldest inventory. In this case, we also observe separation of inventories.

Table 4: Separation of inventories in the presence of a positive transshipment cost

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<td>Y</td>
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<td>Y</td>
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<td>Y</td>
<td>Y</td>
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<tr>
<td></td>
<td>(t = 2)</td>
<td>(\theta = 0)</td>
<td>N</td>
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<tr>
<td></td>
<td>(\theta = 1)</td>
<td>N</td>
<td>N</td>
<td>Y</td>
<td>N</td>
</tr>
<tr>
<td></td>
<td>(\theta = 2)</td>
<td>N</td>
<td>Y</td>
<td>Y</td>
<td>N</td>
</tr>
</tbody>
</table>

Note. \(T = 10\), \(n = 2\), \(r = 10\), \(\alpha = 0.99\), \(D = Uniform[0, 3]\). “Y” and “N” represent separation and no separation, respectively.

One possible challenge of implementing transshipment, as we mentioned earlier, stems from the misaligned incentive of store managers. This is not a major problem in convenience store chains (i.e., 7-Eleven Hong Kong) where one owner owns several outlets in close proximity and replenishment is completely centralized. In the extreme case, the incentive issue will go away if the stores are no longer managed by people. Unmanned stores such as Amazon Go in Seattle and BingoBox in Shanghai have generated a lot of discussion (Dastin 2018 and Soo 2017). It would be fascinating to experiment transshipment in those stores.

Waste is a universal problem. Intense discussions have been made and government regulations implemented to increase the cost of waste disposal or even ban food waste in landfills. In our earlier analysis, we assumed that the retailer incurred an outdating cost for every unit of waste. Alternatively, one can also impose a constraint on the expected amount of waste the retailer can generate each period. Regardless, it is a challenge to strike the right balance between the need to reduce waste and the business and consumer interests. Helping retailers to improve their operations, however, may boost their bottom lines and at the same time reduce waste. This is an area where operations researchers can do more.
Proof of Lemma 1: Since the state variable $x_j$ appears in the linear term $s \sum_{j=1}^{n-1} x_j$ and in the constraint $z_j^1 + z_j^2 \leq x_j$, it is easy to see that $v_t(x + \delta e_j) - v_t(x) \geq s \delta$. This result shows that optimal profit is higher when initial inventory is higher.

(i) The proof is achieved by induction. The result obviously holds for $v_{T+1}(x)$. Suppose
\[ v_{t+1}(x + \delta e_i) \leq v_{t+1}(x + \delta e_{i+1}) \]
for $1 \leq i \leq n - 2$. We first show that $v_t(x + \delta e_1) \leq v_t(x + \delta e_2)$. Let $(\bar{z}^1, \bar{q}^1)$ and $(\bar{z}^2, \bar{q}^2)$ be the optimal solution for the state $x + \delta e_1$. If both $\bar{z}_1^1$ and $\bar{z}_1^2$ equal zero, then $(\bar{z}^1, \bar{q}^1)$ and $(\bar{z}^2, \bar{q}^2)$ are feasible for the state $x + \delta e_2$, and hence obviously $v_t(x + \delta e_2) \geq v_t(x + \delta e_1)$. Suppose either $\bar{z}_1^1$ or $\bar{z}_1^2$ is positive, and without loss of generality, let us assume $\bar{z}_1^1 > 0$. We can construct a new policy $(\bar{z}^1, \bar{q}^1)$ and $(\bar{z}^2, \bar{q}^2)$ where $(\bar{z}^2, \bar{q}^2) = (\bar{z}^2, \bar{q}^2)$, $z_1^1 = \bar{z}_1^1 - \delta$, $z_2^1 = \bar{z}_1^2 + \delta$, $z_j^1 = \bar{z}_j$ and $q_1^1 = \bar{q}^1$ for $j \geq 3$. This new policy is feasible for $x + \delta e_2$, and
\[
v_t(x + \delta e_2) - v_t(x + \delta e_1) \geq \theta E(\bar{q}^1 + \sum_{j=2}^{n-1} z_j^1 + \delta - D^1)^+ - \theta E(\bar{q}^1 + \sum_{j=2}^{n-1} z_j^2 - D^1)^+ + \Delta \\
\geq 0,
\]
where $\Delta = \alpha E v_{t+1}(\sum_{i=1}^{2} Y^i(q^i, z^i, D^i)) - \alpha E v_{t+1}(\sum_{i=1}^{2} Y^i(q^i, \bar{z}^i, D^i))$ is the additional profit from future periods from using the new policy. The second inequality holds since the new policy leads to a higher initial inventory with a one-period lifetime in the next period, and hence the additional profit from future periods $\Delta$ is positive.

To show that $v_t(x + \delta e_i) \leq v_t(x + \delta e_{i+1})$, let $(\bar{z}^1, \bar{q}^1)$ and $(\bar{z}^2, \bar{q}^2)$ be the optimal solution for the state $x + \delta e_i$. If both $\bar{z}_1^1$ and $\bar{z}_1^2$ equal zero, then $(\bar{z}^1, \bar{q}^1)$ and $(\bar{z}^2, \bar{q}^2)$ are feasible for the state $x + \delta e_{i+1}$, and hence obviously $v_t(x + \delta e_{i+1}) \geq v_t(x + \delta e_i)$. If either $\bar{z}_1^1$ or $\bar{z}_1^2$ is positive, we can similarly construct a new policy $(\bar{z}^1, \bar{q}^1)$ and $(\bar{z}^2, \bar{q}^2)$ where $(\bar{z}^2, \bar{q}^2) = (\bar{z}^2, \bar{q}^2)$, $z_1^1 = \bar{z}_1^1 - \delta$, $z_1^2 = \bar{z}_1^2 + \delta$, $z_j^1 = \bar{z}_j$ and $q_1^1 = \bar{q}^1$ for $j \leq i$ and $j > i + 1$. This new policy is feasible for $x + \delta e_{i+1}$, and it leads to a fresher initial inventory in the next period. Hence by the induction hypothesis, we have $v_t(x + \delta e_{i+1}) \geq v_t(x + \delta e_i)$.

(ii) Let $(\bar{z}^1, \bar{q}^1)$ and $(\bar{z}^2, \bar{q}^2)$ be the optimal solution for the state $x + \delta e_{n-1}$. If both $\bar{z}_{n-1}^1$ and $\bar{z}_{n-1}^2$ equal zero, then $(\bar{z}^1, \bar{q}^1)$ and $(\bar{z}^2, \bar{q}^2)$ are feasible for the state $x$, and hence
\[
v_t(x + \delta e_{n-1}) - v_t(x) \leq s \delta \\
\leq c \delta.
\]
The first inequality holds because $v_t(x)$ is greater than or equal to the profit under any feasible policy. Otherwise, let us assume $\bar{z}_{n-1}^1 > 0$. We can construct a new policy $(z^1, q^1)$ and $(z^2, q^2)$
where \((z^2, q^2) = (z^2, q^2), z_{n-1}^1 = z_{n-1}^1 - \delta, q^1 = q + \delta, z_j^1 = z_j^2\) and \(q = q^1\) for \(j < n - 1\). The new policy is feasible for the state \(x\), and

\[
v_t(x + \delta e_{n-1}) - v_t(x) \leq s\delta - s\delta + c\delta + \Delta \\
\leq c\delta,
\]

where \(-\Delta = \alpha E v_{t+1}(\sum_{i=1}^{n} Y(t+1)_{i}^i(q^1, z^i, D^i)) - \alpha E v_{t+1}(\sum_{i=1}^{n} Y(t+1)_{i}^i(\bar{q}^i, \bar{z}^i, D^i))\) is the additional profit from future periods from using the new policy. The second inequality holds since the new policy leads to fresher initial inventories in the next period, and hence \(-\Delta\) is non-negative by part (i).

(iii) The first-order derivative of \(J_t(z^i, q^i)\) with respect to \(z^i_1\) is

\[
\frac{\partial J_t(z^i, q^i)}{\partial z^i_1} = (p - s) - (p + \theta)\Phi(q^i + \sum_{j=1}^{n-1}z^i_j).
\]

The result follows because the derivative is decreasing in \(q^i\) and \(z^i_j\). Furthermore, we can see that \(J_t(z^i, q^i)\) is submodular in \((z^i_1, q^i + \sum_{j=1}^{n-1}z^i_j)\) for any \(l \geq 2\).

**Proof of Theorem 1:** (i) The proof is achieved by contradiction. Suppose for all optimal policies we have \(\bar{q}^1 > 0\) or \(\bar{q}^2 > 0\) or \(\bar{z}^1_j > 0\) or \(\bar{z}^2_j > 0\) for some \(j < i\).

If \(\bar{q}^1 > 0\), we let \(\delta = \min(x_i - \bar{z}^1_i - \bar{z}^2_i, \bar{q}^1) > 0\) and construct a new policy \(z^1 = \bar{z}^1 + \delta e_i\), and \(q^1 = \bar{q}^1 - \delta\). The new policy will lead to an immediate profit increase of \((c - s)\delta\). However, the new policy will result in less fresh initial inventory in the next period, but the loss is smaller than \(\alpha(c - s)\delta\). Hence the new policy will increase the profit, which is a contradiction. Therefore, \(\bar{q}^1 = 0\) and similarly we can show that \(\bar{q}^2 = 0\).

If \(\bar{z}^1_j > 0\) for some \(j < i\), we let \(\delta = \min(x_i - \bar{z}^1_i - \bar{z}^2_i, \bar{z}^1_j) > 0\) and construct a new policy \(z^1 = \bar{z}^1 - \delta e_j + \delta e_i\), and \(q^1 = \bar{q}^1\). The new policy will increase profit by \(\theta(E(\bar{q}^1 + \delta - D^1)^+ - E(\bar{q}^1 - D^1)^+))\) in the current period and will lead to fresher initial inventory in the next period, which is a contradiction. Thus, \(\bar{z}^1_j = 0\) and similarly we can show that \(\bar{z}^2_j = 0\).

(ii) Suppose that \(\sum_{i=2}^{n-1} x_i \geq 2l_0\). If there exists an \(i \geq 2\) such that \(\bar{z}^1_i + \bar{z}^2_i < x_i\), then the result holds according to Theorem 1. Otherwise, we have \(\bar{z}^1_i + \bar{z}^2_i = x_i\) for all \(2 \leq i \leq n - 1\). Because \(\sum_{i=2}^{n-1} x_i \geq 2l_0\), we have \(\sum_{i=2}^{n-1} \bar{z}^1_i + \sum_{i=2}^{n-1} \bar{z}^2_i \geq 2l_0\), which means either \(\sum_{i=2}^{n-1} \bar{z}^1_i \geq l_0\) or \(\sum_{i=2}^{n-1} \bar{z}^2_i \geq l_0\). The derivative of \(J_t(z^i, q^i)\) with respect to \(z^i_1\) is

\[
(p + \theta)(\Phi(l_0) - \Phi(\bar{z}^1_1 + \sum_{j=2}^{n-1}z^i_j + q^i)),
\]

which is negative if \(\sum_{j=2}^{n-1}z^i_j \leq l_0\). Therefore, when \(\sum_{j=2}^{n-1}z^i_j \geq l_0\), either \(\bar{z}^1_1\) or \(\bar{z}^2_1\) is zero. □
Proof of Theorem 2: The proof is by contradiction. For ease of exposition, we prove the result for \( n = 3 \). The analysis can be easily extended to \( n > 3 \). Suppose that in all optimal policies, \( \bar{z}_1 > 0, \bar{z}_2 > 0, \bar{q}_1 > 0, \) and \( \bar{q}_2 > 0 \). Without loss of generality, assume \( \bar{q}_1 \geq \bar{q}_2 \).

Let \( \bar{z}_2 = \bar{z}_2^2 = \bar{z}_2^1 \). We construct a new policy:

\[
\begin{align*}
    z_1^1 &= \bar{z}_1^1 - \delta, \quad q_1^1 = \bar{q}_1^1 + \delta, \quad z_2^1 = \bar{z}_2^1 = \bar{z}_2, \\
    z_1^2 &= \bar{z}_1^2 + \delta, \quad q_1^2 = \bar{q}_1^2 - \delta, \quad z_2^2 = \bar{z}_2^2 = \bar{z}_2.
\end{align*}
\]

It is not difficult to see that the new policy is feasible as long as \( 0 \leq \delta \leq \min\{\bar{z}_1^1, \bar{q}_2^2\} \). Let \( \bar{y}_i = \bar{z}_i^2 + \bar{q}_i \) for \( i = 1, 2 \). Then \( \bar{y}_1 \geq \bar{y}_2 \). The value function under the new policy is given by

\[
\begin{align*}
    &\sum_{i=1}^{n-1} s(x_i - \bar{z}_i^1 - \bar{z}_i^2) - c(\bar{q}_1^1 + \bar{q}_2^2) \\
    &+ pE\min\{ \bar{z}_1^1 + \bar{y}_1, D_1^1 \} + pE\min\{ \bar{z}_2^1 + \bar{y}_2, D_2^1 \} \\
    &- \left[ \theta E(\bar{z}_1^1 + \bar{y}_1 - D_1^1)^+ + \theta E(\bar{z}_2^1 + \bar{y}_2 - D_2^1)^+ - \theta E(\bar{y}_1 + \delta - D_1^1)^- - \theta E(\bar{y}_2 - \delta - D_2^1)^+ \right] \\
    &+ \alpha f(\delta),
\end{align*}
\]

where

\[
f(\delta) = E_{v_{t+1}}((\bar{y}_1 + \delta - D_1^1)^+ + (\bar{y}_2 - \delta - D_2^1)^+ - (\bar{q}_1^1 + \delta - D_1^1)^+ - (\bar{q}_2^2 - \delta - D_2^1)^+),\]

The new policy leads to the same total ordering cost and total revenue from regular and clearance sales in the current period as those under the optimal policy. The total expected outdating cost under the new policy is decreasing in \( \delta \) because its derivative with respect to \( \delta \) is equal to

\[
\theta [\Phi(\bar{y}_2 - \delta) - \Phi(\bar{y}_1 + \delta)],
\]

which is negative for \( \delta \geq 0 \).

Let \( V_t \) denote the partial derivative \( \frac{\partial v_{t+1}}{\partial x_i} \). Let \( \bar{\Phi} = 1 - \Phi \). We can show that the first order
implies monotone likelihood ratio, that is, $\phi_0$. By Lemma 1, we know that $\Delta \geq 0$. Note that for $0 \leq x_1 \leq \tilde{z}_2$ and $0 \leq x_2 \leq \tilde{q}_2 - \delta$, we always have $\tilde{y}_2 - \delta - x_1 \geq \tilde{q}_2 - \delta - x_2$. Because the log-concavity of $\phi$ implies monotone likelihood ratio, that is, $\phi(x - \theta_2)/\phi(x - \theta_1)$ is increasing in $x$ for any $\theta_1 < \theta_2$, we have

$$\frac{\phi(\tilde{y}_2 - \delta - x_1)}{\phi(\tilde{y}_1 + \delta - x_1)} = \frac{\phi(\tilde{y}_2 - \delta - x_1)}{\phi(\tilde{y}_2 - \delta - x_2)} \geq \frac{\phi(\tilde{q}_2 - \delta - x_2)}{\phi(\tilde{q}_1 + \delta - x_2)}.$$

Thus, $\tilde{\Delta} \geq 0$. Log-concavity of density function also implies increasing failure rate, so we have

$$\frac{\phi(\tilde{q}_1 + \delta - x)}{\Phi(\tilde{q}_1 + \delta - x)} \geq \frac{\phi(\tilde{q}_2 - \delta - x)}{\Phi(\tilde{q}_2 - \delta - x)}.$$
Furthermore, log-concavity of $\phi(x)$ implies that $\Phi(x)$ is also log-concave and
\[
y^1 + \delta - (q^1 + \delta - x) = y^2 - \delta - (q^2 - \delta - x) = \bar{z}_2 + x,
\]
we have
\[
\log \Phi(y^1 + \delta) - \log \Phi(q^1 + \delta - x) \leq \log \Phi(y^2 - \delta) - \log \Phi(q^2 - \delta - x).
\]
Thus,
\[
\frac{\Phi(q^1 + \delta - x)}{\Phi(y^1 + \delta)} \geq \frac{\Phi(q^2 - \delta - x)}{\Phi(y^2 - \delta)},
\]
and we have
\[
\frac{\phi(q^1 + \delta - x)}{\Phi(y^1 + \delta)} \geq \frac{\phi(q^2 - \delta - x)}{\Phi(y^2 - \delta)}.
\]
We can similarly show
\[
\frac{\phi(y^1 + \delta - x)}{\Phi(y^1 + \delta)} \geq \frac{\phi(y^2 - \delta - x)}{\Phi(y^2 - \delta)}.
\]
Therefore, we can conclude that $f(\delta)$ is increasing in $\delta$ and so is the value function. This means that there exists an optimal solution where either $z^1 = 0$ or $q^2 = 0$, which is a contradiction. \(\square\)

**Proof of Theorem 3:** Let $\bar{y}^i = \sum_{j=2}^{n-1} z_j^i + q^i$ for $i = 1, 2$. Without loss of generality, suppose in all optimal policies, $z_1 > 0$, $z_2 > 0$ and $z_1^1 + \bar{y}^1 > z_1^2 + \bar{y}^2$. Consider a new policy with $z_1^1 = z_1^1 - \delta$, $z_2^1 = z_2^1 + \delta$, $q^1 = q^1$, $q^2 = q^2$ and $z_j^k = z_j^k$ for $j \geq 2$ and $k = 1, 2$. The new policy is feasible as long as $0 \leq \delta \leq \bar{z}_1^1$.

It is easy to see that the value function under new policy is given by
\[
\sum_{i=1}^{n-1} s(x_i - z_i^1 - z_i^2) - c(q^1 + q^2) + pE \min \{z_1^1 + \bar{y}^1 - \delta, D^1\} + pE \min \{z_2^1 + \bar{y}^2 + \delta, D^2\} - \theta E(z_1^1 - \delta - (D^1 - \bar{y}^1)^+) + \theta E(z_2^1 + \delta - (D^2 - \bar{y}^2)^+) + \alpha E_{t+1} \left( \sum_{i=1}^{2} Y^i(q^1, z^1, D^i) \right).
\]
Its first order derivative with respect to $\delta$ is
\[
(p + \theta)[\Phi(z_1^1 + \bar{y}^1 - \delta) - \Phi(z_2^1 + \bar{y}^2 + \delta)],
\]
which is greater than zero as long as $z_1^1 + \bar{y}^1 - \delta \geq z_1^2 + \bar{y}^2 + \delta$. This means that we can find a new policy that is better than the optimal policy and either $z_1^1 = 0$ or $z_1^1 + y^1 = z_2^1 + y^2$, which is a contradiction. \(\square\)

**Proof of Lemma 2:** (i) For any $D^i$,
\[
(z_1^i - (D^i - y^i)^+) = (z_1^i + y^i - D^i)^+ - (y^i - D^i)^+.
\]
and
\[ \min(D^i, z_i^1 + y^i) = z_i^1 + y^i - (z_i^1 + y^i - D^i)^+ . \]

So
\[ J_t(z_i^1, y^i) = (p - s)z_i^1 + (p - c)y^i - (p + \theta)E(z_i^1 + y^i - D^i)^+ + (\theta + v)E(y^i - D^i)^+ . \]

The first, second and last terms depend on only one variable, and they are hence submodular. The third is a concave function of \( z_i^1 + y^i \) and is therefore submodular in \((z_i^1, y^i)\) (Lemma 2.6.2, Topkis 1998). The result follows because the sum of submodular functions is still submodular.

(ii) Let
\[ f^i(y) = (p - s)z_i^1 + (p - c)y + (p - c)\mu^i - (p + \theta)(z_i^1 + y)^+ + (\theta + v)y^+ . \]

Then \( J_t(z_i^1, y^i) = \mathbb{E} f^i(y^i - D^i) \). It is easy to show that \( f^i(y) \) is quasiconcave; it is first increasing, then decreasing, and finally decreasing but with a more gentle slope. The result hence follows.

\[ \square \]

**Proof of Theorem 4:** The proof is achieved by contradiction. Suppose that in all optimal policies, \( \bar{z}_i^1 > 0, \bar{z}_i^2 > 0, \bar{y}_1 > 0, \) and \( \bar{y}_2 > 0 \). Without loss of generality, assume \( \bar{y}_1 \geq \bar{y}_2 \).

Let \( \delta = \min(\bar{z}_i^1, \bar{y}_2) \) and we construct a new policy:
\[
\begin{align*}
    z_i^1 &= \bar{z}_i^1 - \delta, \\
    z_i^2 &= \bar{z}_i^2 + \min(\delta, \bar{z}_i^2), \\
    y^1 &= \bar{y}_1 + \delta, \\
    y^2 &= \bar{y}_2 - \delta.
\end{align*}
\]

It is not difficult to show that the new policy is still feasible and either \( z_i^1 \) or \( y^2 \) is zero and \( z_i^1, z_i^2, y^1, z_i^2, z_i^2, y^2 \) are all nonnegative. The objective function \( J_t \) can be written as
\[
J_t(z_i^1, y^i) = -sz_i^1 - cy^i + p\mathbb{E} \min(D^i, z_i^1 + y^i) - \theta \mathbb{E}(z_i^1 + y^i - D^i)^+ + (\theta + \alpha v)\mathbb{E}(y^i - D^i)^+ .
\]

The expected profit under the new policy minus that under the optimal policy is
\[
(\theta + \alpha v)[\mathbb{E}(\bar{y}_1 + \delta - D^i)^+ + \mathbb{E}(\bar{y}_2 - \delta - D^i)^+ - \mathbb{E}(\bar{y}_1 - D^i)^+ - \mathbb{E}(\bar{y}_2 - D^i)^+] .
\]

Because the function \( \mathbb{E}(x - D^i)^+ \) is a convex function, the above expression is positive. This means that the new policy is also optimal, which is a contradiction.

(i) When \( \bar{z}_i^1 + \bar{z}_i^2 = 0, \)
\[
v_t(x_1, x_2) = s(x_1 + x_2) + \max_{\bar{z}_i^1 + \bar{z}_i^2 \leq x_1, x_2 \geq 0} \{ (c - s)(\bar{z}_i^1 + \bar{z}_i^2) + J_t(0, \bar{z}_i^1) + J_t(0, \bar{z}_i^2) \} .
\]

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The function \((c - s)z + J_t(0, z)\) is concave in \(z\). Suppose there is an optimal solution such that \(\tilde{z}_1 \neq \tilde{z}_2\). Then, the symmetric solution \((\frac{\tilde{z}_1 + \tilde{z}_2}{2}, \frac{\tilde{z}_1 + \tilde{z}_2}{2})\) is also an optimal solution.

(ii) follows from Theorem 3. (iii) is obvious. □

Throughout the rest of the appendix, we use the notations \(\tilde{z}_k^i(x_1, x_2|K_i)\) and \(\tilde{y}_i^j(x_1, x_2|K_i)\) to denote the optimal solutions to the optimization problem \(K_i\) in (4). The following lemma characterizes the monotonicity of the optimal solution to each of the three maximization problems in (4).

**Lemma 3**

(i) The function \(\tilde{z}_1^1(x_1, x_2|K_1)\) is increasing in \(x_1\) and decreasing in \(x_2\), and \(\tilde{y}_1^1(x_1, x_2|K_1)\) is decreasing in \(x_1\) and increasing in \(x_2\);

(ii) The function \(\tilde{z}_1^2(x_1, x_2|K_2)\) is increasing in \(x_1\) and decreasing in \(x_2\), and \(\tilde{y}_2^2(x_1, x_2|K_2)\) is decreasing in \(x_1\) and increasing in \(x_2\).

(iii) If \(x_2 < 2u_0\), \(K_3(x_1, x_2) \leq K_2(x_1, x_2)\).

**Proof of Lemma 3**: (i) We first look at the optimization problem \(K_1\). Let \(\tilde{y}_1^1 = -y^1\), \(\tilde{x}_2 = -x_2\) and \(x_1 - \tilde{z}_1^2 = \tilde{z}_2^2\). Then the objective function is supermodular in \((z_1^1, \tilde{z}_1^2, \tilde{y}_1^1, x_1, \tilde{x}_2)\) and the constraint set forms a lattice. Therefore, \(\tilde{z}_1^1(x_1, x_2|K_1)\) is increasing in \(x_1\) and decreasing in \(x_2\), and \(\tilde{y}_1^1(x_1, x_2|K_1)\) is decreasing in \(x_1\) and increasing in \(x_2\).

(ii) We next look at the optimization problem \(K_2\). Let \(\tilde{y}_1^1 = x_2 - y^1\), \(\tilde{x}_1 = -x_1\) and \(\tilde{z}_1^2 = -z_1^2\). Then the objective function is supermodular in \((\tilde{z}_1^2, \tilde{y}_1^1, y_2, \tilde{x}_1, x_2)\) and the constraint set forms a lattice. Therefore, \(\tilde{y}_1^1(x_1, x_2|K_2)\) is decreasing in \(x_1\) and increasing in \(x_2\), and \(\tilde{z}_1^2(x_1, x_2|K_2)\) is increasing in \(x_1\) and decreasing in \(x_2\).

(iii) Note first that \(u_0 = \arg \max_{z \geq 0} \{(c - s)z + J_t(0, z)\}\). When \(x_2 < 2u_0\), the optimal solution to \(K_3\) must be a boundary solution; that is, \(\tilde{z}_1^2(x_1, x_2|K_3) + \tilde{z}_2^1(x_1, x_2|K_3) = x_2\). It is easy to see that the optimal solution to \(K_3\) is a feasible but not necessarily optimal solution to \(K_2\). □

**Proof of Theorem 5**: The proof of the theorem is based on the optimization problem (4).

(i) For \(x_2 \geq 2u_0\), if we can show that \(K_1 \leq K_2 \leq K_3\), then the result follows. We can simplify the optimization problem \(K_1\) by sequentially optimizing the objective function with respect to each decision variable, first \(z_1^2\), then \(z_1^1\), and finally \(y^1\). Note that

\[
\frac{\partial J_t(z, y)}{\partial z} = p - s - (p + \theta)\Phi(z + y) = (p + \theta)(\Phi(l_0) - \Phi(z + y)).
\]
We obtain the following two cases after optimizing over $z_1^2$.

Case 1: If $x_1 \leq l_0$,

$$K_1 - (c - s)x_2 = \max_{0 \leq z_i^2 \leq x_1 - z_i^1, y_i^1 \geq x_2, z_i^1 \geq 0} \{J_t(z_1^1, y_1^1) + J_t(z_2^1, 0)\}$$

$$= \max_{y_i^1 \geq x_2, 0 \leq z_i^1 \leq x_1} \{J_t(z_1^1, y_1^1) + J_t(x_1 - z_1^1, 0)\}.$$

Based on the following derivative,

$$\frac{\partial}{\partial z_1^1} [J_t(z_1^1, y_1^1) + J_t(x_1 - z_1^1, 0)] = \max \left( p + \theta (\Phi(x_1 - z_1^1) - \Phi(z_1^1 + y_1^1)) \right),$$

we know that $\arg \max_{z_1^1 \geq 0} J_t(z_1^1, y_1^1) + J_t(x_1 - z_1^1, 0) = (x_1 - y_1^1)^+ / 2$. There are two sub-cases after optimizing over $z_1^1$.

Case 1(i): If $x_2 \leq x_1$,

$$K_1 - (c - s)x_2 = \max_{y_i^1 \geq x_1} \{\max_{x_2 \leq y_i^1 \leq x_1} \{J_t(0, y_1^1) + J_t(x_1, 0)\}, \max_{x_2 \leq y_i^1 \leq x_1} \{J_t(x_1 - y_1^1/2, y_1^1) + J_t(x_1 + y_1^1/2, 0)\}\}.$$

Case 1(ii) If $x_2 > x_1$,

$$K_1 - (c - s)x_2 = \max_{y_i^1 \geq x_1} \{J_t(0, y_1^1) + J_t(x_1, 0)\}.$$

Case 2: If $x_1 > l_0$,

$$K_1 - (c - s)x_2 = \max_{0 \leq z_i^2 \leq x_1 - z_i^1, y_i^1 \geq x_2, z_i^1 \geq 0} \{J_t(z_1^1, y_1^1) + J_t(z_2^1, 0)\}$$

$$= \max_{x_1 - z_1^1 \geq 0, y_i^1 \geq x_2, z_i^1 \geq 0} \{J_t(z_1^1, y_1^1) + J_t(l_0, 0)\},$$

$$\max_{x_1 - z_1^1 < 0, y_i^1 \geq x_2, z_i^1 \geq 0} \{J_t(z_1^1, y_1^1) + J_t(x_1 - z_1^1, 0)\}.$$

Optimizing over $z_1^1$, we have the following three sub-cases.

Case 2(i): If $x_2 \leq 2l_0 - x_1$,

$$K_1 - (c - s)x_2 = \max_{2l_0 - x_1 \leq y_i^1 \leq l_0} \{\max_{2l_0 - x_1 \leq x_2 \leq y_i^1 \leq 2l_0 - x_1} \{J_t(l_0 - y_1^1, y_1^1) + J_t(l_0, 0)\}, \max_{x_2 \leq y_i^1 \leq 2l_0 - x_1} \{J_t(x_1 - l_0, y_1^1) + J_t(l_0, 0)\}, \max_{y_i^1 \geq x_2} \{J_t(x_1 - l_0, y_1^1) + J_t(l_0, 0)\}\}.$$

$$\max_{y_i^1 \geq x_2} \{J_t(x_1 - l_0, y_1^1) + J_t(l_0, 0)\},$$

$$\max_{x_2 \leq y_i^1 \leq 2l_0 - x_1} \{J_t(x_1 - y_1^1/2, y_1^1) + J_t(x_1 + y_1^1/2, 0)\}\}.$$

The second equality holds since $J_t(l_0 - y_1^1, y_1^1)$ is convex in $y_1^1$ and its derivative is

$$\frac{\partial}{\partial y_1^1} [J_t(l_0 - y_1^1, y_1^1)] = s - c + (\theta + \alpha v)\Phi(y_1^1).$$
Case 2(ii): If \( 2l_0 - x_1 \leq x_2 \leq l_0 \),

\[
K_1 - (c - s)x_2 = \max_{y^1 \geq x_2} \{ J_l(0, y^1) + J_t(l_0, 0) \}, \max_{y^1 \geq x_2} \{ J_l(0, y^1) + J_t(l_0, 0) \}, \max_{y^1 \geq x_2} \{ J_l(x_1 - l_0, y^1) + J_t(l_0, 0) \}, \max_{y^1 \geq x_2} \{ J_l(x_1 - l_0, y^1) + J_t(l_0, 0) \}).
\]

Case 2(iii): If \( x_2 \geq l_0 \),

\[
K_1 - (c - s)x_2 = \max_{y^1 \geq x_2} \{ J_l(0, y^1) + J_t(l_0, 0) \}, \max_{y^1 \geq x_2} \{ J_l(x_1 - l_0, y^1) + J_t(l_0, 0) \}).
\]

From the above discussion, we know that for \( x_2 \geq 2u_0 \geq 2l_0 \), either we have Case 2(iii) where

\[
K_1 - (c - s)x_2 = \max_{y^1 \geq x_2} \{ J_l(0, y^1) + J_t(l_0, 0) \}, \max_{y^1 \geq x_2} \{ J_l(x_1 - l_0, y^1) + J_t(l_0, 0) \}),
\]

or we have Case 1(ii) where

\[
K_1 - (c - s)x_2 = \max_{y^1 \geq x_2} \{ J_l(0, y^1) + J_t(l_0, 0) \}.
\]

Since \( J_t(0, y^1) \leq J_t(x_1 - l_0, y^1) \) for any \( y^1 \geq x_2 \geq 2u_0 \), we have \( \bar{z}^1(x_1, x_2|K_1) = 0 \) and \( K_1 \leq K_2 \).

We can similarly simplify the optimization problem \( K_2 \) by sequentially optimizing the objective function with respect to each decision variable, first \( y^1 \), then \( z^2_1 \), and finally \( y^2 \). Note that

\[
\frac{\partial J_t(0, y)}{\partial y} = p - c - (p - \alpha v) \Phi(y)
\]

\[
= (p - \alpha v)(\Phi(u) - \Phi(y)).
\]

There are two cases after optimizing over \( y^1 \).

Case 1: If \( x_2 \leq u \),

\[
K_2 - (c - s)x_2 = \max_{0 \leq z^2 \leq x_1, y^1 \geq x_2, y^2 \geq 0} \{ J_l(0, y^1) + J_t(z^2, y^2) \}
\]

Optimizing over \( z^2_1 \), we further obtain two sub-cases.

Case 1(i): If \( x_1 \geq l_0 \),

\[
K_2 - (c - s)x_2 = \max_{0 \leq z^2 \leq x_1, y^2 \geq 0} \{ J_l(0, u) + J_t(z^2, y^2) \}
\]

\[
= \max_{0 \leq y^2 \leq l_0} \{ J_l(0, u) + J_t(l_0 - y^2, y^2) \}, \max_{y^2 \geq l_0} \{ J_l(0, u) + J_t(0, y^2) \}.
\]
Case 1(ii): If $x_1 \leq l_0$,

$$K_2 - (c-s)x_2 = \max_{0 \leq z_1^2 \leq x_1, y^2 \geq 0} \{J_t(0, u) + J_t(z_1^2, y^2)\}$$

$$= \max \{\max_{l_0 - x_1 \leq y^2 \leq l_0} \{J_t(0, u) + J_t(l_0 - y^2, y^2)\}, \max_{0 \leq y^2 \leq l_0 - x_1} \{J_t(0, u) + J_t(x_1, y^2)\}\} \max_{y^2 \geq 0} \{J_t(0, u) + J_t(0, y^2)\}.$$ 

Case 2: If $x_2 \geq u$,

$$K_2 - (c-s)x_2 = \max_{0 \leq z_1^2 \leq x_1, 0 \leq y^2 \leq x_2 - u} \{J_t(0, y^1) + J_t(z_1^2, y^2)\}$$

$$= \max \{\max_{0 \leq y^2 \leq x_2 - u} \{J_t(0, x_2 - y^2) + J_t(z_1^2, y^2)\}, \max_{0 \leq x_1, 0 \leq y^2 \leq x_2 - u} \{J_t(0, u) + J_t(x_1, y^2)\}\} \max_{x_2 - u \leq y^2 \leq x_2 - u} \{J_t(0, u) + J_t(0, y^2)\}.$$ 

Optimizing over $z_1^2$, we obtain the following five sub-cases.

Case 2(i): If $x_1 \geq l_0$ and $x_2 - u \geq l_0$,

$$K_2 - (c-s)x_2 = \max \{\max_{0 \leq y^2 \leq x_2 - u} \{J_t(0, x_2 - y^2) + J_t(z_1^2, y^2)\}, \max_{0 \leq y^2 \leq l_0} \{J_t(0, u) + J_t(z_1^2, y^2)\}\} \max_{x_2 - u \leq y^2 \leq x_2 - u} \{J_t(0, u) + J_t(0, y^2)\}.$$ 

Case 2(ii): If $x_1 \geq l_0$ and $x_2 - u \leq l_0$

$$K_2 - (c-s)x_2 = \max \{\max_{0 \leq y^2 \leq x_2 - u} \{J_t(0, x_2 - y^2) + J_t(z_1^2, y^2)\}, \max_{0 \leq y^2 \leq x_2 - u} \{J_t(0, u) + J_t(z_1^2, y^2)\}\} \max_{x_2 - u \leq y^2 \leq x_2 - u} \{J_t(0, u) + J_t(0, y^2)\}.$$ 

Case 2(iii): If $x_1 \leq l_0$ and $l_0 - x_1 \geq x_2 - u$

$$K_2 - (c-s)x_2 = \max \{\max_{0 \leq y^2 \leq x_2 - u} \{J_t(0, x_2 - y^2) + J_t(z_1^2, y^2)\}, \max_{0 \leq y^2 \leq x_2 - u} \{J_t(0, u) + J_t(z_1^2, y^2)\}\} \max_{x_2 - u \leq y^2 \leq x_2 - u} \{J_t(0, u) + J_t(0, y^2)\}.$$ 

$$= \max \{\max_{0 \leq y^2 \leq x_2 - u} \{J_t(0, x_2 - y^2) + J_t(z_1^2, y^2)\}, \max_{0 \leq y^2 \leq x_2 - u} \{J_t(0, u) + J_t(x_1, y^2)\}\} \max_{x_2 - u \leq y^2 \leq x_2 - u} \{J_t(0, u) + J_t(0, y^2)\}.$$
Case 2(iv): If $x_1 \leq l_0$ and $l_0 - x_1 \leq x_2 - u \leq l_0$

$$K_2 - (c-s)x_2 = \max\{\max_{0 \leq y^2 \leq l_0 - x_1} \{J_t(0, x_2 - y^2) + J_t(l_0 - y^2, y^2)\}, \max_{0 \leq y^2 \leq l_0 - x_1} \{J_t(0, x_2 - y^2) + J_t(x_1, y^2)\}, \max_{0 \leq y^2 \leq l_0} \{J_t(0, u) + J_t(l_0 - y^2, y^2)\}, \max_{y^2 \geq l_0} \{J_t(0, u) + J_t(l_0, y^2)\}\}.$$ 

Case 2(v): If $x_1 \leq l_0$ and $x_2 - u \geq l_0$,

$$K_2 - (c-s)x_2 = \max\{\max_{0 \leq y^2 \leq l_0 - x_1} \{J_t(0, x_2 - y^2) + J_t(l_0 - y^2, y^2)\}, \max_{0 \leq y^2 \leq l_0 - x_1} \{J_t(0, x_2 - y^2) + J_t(x_1, y^2)\}, \max_{y^2 \geq l_0} \{J_t(0, u) + J_t(l_0, y^2)\}\}.$$ 

From the above discussion, we know that for $x_2 \geq 2u_0$, either we have Case 2(i) where

$$K_2 - (c-s)x_2 = \max\{\max_{0 \leq y^2 \leq l_0} \{J_t(0, x_2 - y^2) + J_t(l_0 - y^2, y^2)\}, \max_{0 \leq y^2 \leq l_0} \{J_t(0, x_2 - y^2) + J_t(0, y^2)\}, \max_{y^2 \geq x_2 - u} \{J_t(0, u) + J_t(l_0, y^2)\}\},$$

or we have Case 2(v) where

$$K_2 - (c-s)x_2 = \max\{\max_{0 \leq y^2 \leq l_0} \{J_t(0, x_2 - y^2) + J_t(l_0 - y^2, y^2)\}, \max_{0 \leq y^2 \leq l_0 - x_1} \{J_t(0, x_2 - y^2) + J_t(x_1, y^2)\}, \max_{y^2 \geq x_2 - u} \{J_t(0, u) + J_t(l_0, y^2)\}\}.$$ 

Because for $0 \leq y^2 \leq l_0 - x_1$,

$$\frac{\partial}{\partial y^2} \{J_t(0, x_2 - y^2) + J_t(x_1, y^2)\} = (p - \alpha v)\Phi(x_2 - y^2) - (p + \theta)\Phi(x_1 + y^2) + (\theta + \alpha v)\Phi(y^2)$$

$$\geq (p - \alpha v)\Phi(x_2 - y^2) - (p + \theta)\Phi(x_1 + y^2)$$

$$\geq (p - \alpha v)\Phi(x_2 + x_1 - l_0) - (p + \theta)\Phi(l_0)$$

$$\geq 0$$

and for $0 \leq y^2 \leq l_0$,

$$\frac{\partial}{\partial y^2} \{J_t(0, x_2 - y^2) + J_t(l_0 - y^2, y^2)\} = -(p - s) + (p - \alpha v)\Phi(x_2 - y^2) + (\theta + \alpha v)\Phi(y^2)$$

$$\geq -(p - s) + (p - \alpha v)\Phi(x_2 - l_0)$$

$$\geq 0,$$

we can simplify $K_2$ as

$$K_2 - (c-s)x_2 = \max_{l_0 \leq y^2 \leq x_2 - u} \{J_t(0, x_2 - y^2) + J_t(0, y^2)\} = J_t(0, x_2/2) + J_t(0, x_2/2).$$
Thus, we have $K_2 \leq K_3$.

(ii) According to Lemma 3(i) and (ii), we can define $A_1(x_2) = \max\{x_1 : z_1^i(x_1, x_2|K_1) = 0\}$ and $A_2(x_2) = \max\{x_1 : z_1^2(x_1, x_2|K_2) = 0\}$. Both $A_1(x_2)$ and $A_2(x_2)$ are increasing in $x_2$. Let $A(x_2) = \min\{A_1(x_2), A_2(x_2)\}$, then it is easy to see that it is optimal clear all old inventory if $x_1 \leq A(x_2)$. □

We introduce the following result which will be used in later proofs.

**Lemma 4** If $u \leq l_0$, then $J_t(l_0, 0) \geq J_t(0, u)$.

**Proof of Lemma 4:** The inequality $u \leq l_0$ is equivalent to $\frac{c-s}{\theta + av} \geq \Phi(l_0)$. Because $\frac{\partial J_t(x, 0)}{\partial x} \geq \frac{\partial J_t(0, x)}{\partial x}$ for $\Phi(x) \leq \frac{c-s}{\theta + av}$, we have $J_t(l_0, 0) \geq J_t(0, u)$. □

**Proof of Theorems 6, 7, and 8:** Since Theorems 6, 7, and 8 focus on different aspects of the optimal policy, here we provide a unified proof by first characterizing the optimal policy under different states and then defining appropriate switching curves $B(x_1)$ and $C(x_1)$.

Let $O(x_1) = \arg \max_{y_1 \geq 0} J_t(x_1, y_1)$ and $l = \Phi^{-1}(\frac{c-s}{\theta + av})$. It is easy to show that $O(x_1)$ is decreasing and $O(x_1) = 0$ for $x_1 \geq l$. Moreover, $O(x_1) \leq u$. Based on the values of the parameters, the proof below is divided into three parts. In Part I, $u \leq l_0$. In Part II, $u > l_0$ and $J_t(l_0, 0) \leq J_t(0, u)$. In Part III, $u > l_0$ and $J_t(l_0, 0) > J_t(0, u)$.

**Part I:** From the proof of Theorem 5, we can characterize the optimal policy for the optimization problem $K_1$ as follows.

Case 1(i): If $x_1 \leq l_0$ and $x_2 \leq x_1$,

$$K_1 - (c-s)x_2 = \max\{\max_{y_1 \geq x_1} \{J_t(0, y_1) + J_t(x_1, 0)\}, \max_{x_2 \leq y_1 \leq x_1} \{J_t(\frac{x_1 - y_1}{2}, y_1) + J_t(\frac{x_1 + y_1}{2}, 0)\}\}.$$ 

In this case, when $x_1 \leq u$, $K_1(x_1, x_2) = (c-s)x_2 + J_t(0, u) + J_t(x_1, 0)$; and when $x_1 \geq u$, $K_1$ can be simplified as:

$$K_1(x_1, x_2) = (c-s)x_2 + \max_{x_2 \leq y_1 < x_1} \{J_t(\frac{x_1 - y_1}{2}, y_1) + J_t(\frac{x_1 + y_1}{2}, 0)\}.$$ 

Note that here the optimal $y_1 < x_1$.

Case 1(ii): If $x_1 \leq l_0$ and $x_2 > x_1$,

$$K_1 - (c-s)x_2 = \max_{y_1 \geq x_2} \{J_t(0, y_1) + J_t(x_1, 0)\}$$

$$= J_t(0, \max\{x_2, u\}) + J_t(x_1, 0).$$
Similarly, we can characterize the optimal policy for the optimization problem $K_2$.

Case 2(i): If $x_1 \geq l_0$ and $x_2 \leq 2l_0 - x_1$,

$$K_1 - (c - s)x_2 = \max \{ J_t(0, l_0) + J_t(l_0, 0), \max_{y^1 \geq x_2} \{ J_t(x_1 - l_0, y^1) + J_t(l_0, 0) \} \},$$

$$= \max_{x_2 \leq y^1 \leq 2l_0 - x_1} \{ J_t\left(\frac{x_1 - y^1}{2}, y^1 \right) + J_t\left(\frac{x_1 + y^1}{2}, 0 \right) \}$$

$$= J_t(x_1 - l_0, \max\{ x_2, O(x_1 - l_0) \}) + J_t(l_0, 0)$$

Case 2(ii): If $x_1 \geq l_0$ and $2l_0 - x_1 \leq x_2 \leq l_0$,

$$K_1 - (c - s)x_2 = \max \{ \max_{x_2 \leq y^1 \leq l_0} \{ J_t(x_1 - l_0, y^1) + J_t(l_0, 0) \}, \max_{y^1 \geq x_2} \{ J_t(0, y^1) + J_t(l_0, 0) \} \}$$

$$= J_t(l_0 - x_2, x_2) + J_t(l_0, 0).$$

Case 2(iii): If $x_1 \geq l_0$ and $x_2 \geq l_0$,

$$K_1 - (c - s)x_2 = \max \{ \max_{y^2 \geq x_2} \{ J_t(x_1 - l_0, y^1) + J_t(l_0, 0) \}, \max_{y^1 \geq x_2} \{ J_t(0, y^1) + J_t(l_0, 0) \} \}$$

$$= J_t(0, x_2) + J_t(l_0, 0).$$

Similarly, we can characterize the optimal policy for the optimization problem $K_2$.

Case 1(i): If $x_2 \leq u$ and $x_1 \geq l_0$,

$$K_2 - (c - s)x_2 = \max_{0 \leq z_1^2 \leq l_0} \{ J_t(0, u) + J_t(z_1^2, y^2) \}$$

$$= \max \{ \max_{0 \leq y^2 \leq l_0} \{ J_t(0, u) + J_t(l_0 - y^2, y^2) \}, \max_{0 \leq y^2 \leq l_0} \{ J_t(0, u) + J_t(0, y^2) \} \}$$

$$= J_t(0, u) + J_t(l_0, 0).$$

Case 1(ii): If $x_2 \leq u$ and $x_1 \leq l_0$,

$$K_2 - (c - s)x_2 = \max_{0 \leq z_1^2 \leq x_1, y^2 \geq 0} \{ J_t(0, u) + J_t(z_1^2, y^2) \}$$

$$= \max \{ \max_{l_0 - x_1 \leq y^2 \leq l_0} \{ J_t(l_0, u) + J_t(l_0 - y^2, y^2) \}, \max_{0 \leq y^2 \leq l_0 - x_1} \{ J_t(0, u) + J_t(x_1, y^2) \} \}$$

$$= J_t(0, u) + J_t(x_1, O(x_1)).$$

Case 2(i): If $x_1 \geq l_0$ and $x_2 - u \geq l_0$,

$$K_2 - (c - s)x_2 = \max \{ \max_{0 \leq z_1^2 \leq x_1, 0 \leq y^2 \leq x_2 - u} \{ J_t(0, x_2 - y^2) + J_t(z_1^2, y^2) \}, \max_{0 \leq z_1^2 \leq x_1, y^2 \geq x_2 - u} \{ J_t(0, u) + J_t(z_1^2, y^2) \} \}$$

$$= \max \{ \max_{0 \leq y^2 \leq l_0} \{ J_t(l_0, 0) + J_t(0, y^2) \}, \max_{0 \leq y^2 \leq l_0 - u} \{ J_t(l_0, 0) + J_t(0, y^2) \}, \max_{0 \leq y^2 \leq x_2 - u} \{ J_t(0, u) + J_t(0, y^2) \} \}$$

$$= J_t(0, x_2) + J_t(l_0, 0).$$
Case 2(ii): If $x_1 \geq l_0$ and $u \leq x_2 \leq u + l_0$

\[
K_2 - (c - s)x_2 = \max \left\{ \max_{0 \leq y^2 \leq x_2 - u} \{ J_t(0, x_2 - y^2) + J_t(z_1^2, y^2) \}, \max_{0 \leq y^2 \leq l_0 - x_1} \{ J_t(0, u) + J_t(z_1^2, y^2) \} \right\}
\]

\[
= \max \left\{ \max_{0 \leq y^2 \leq x_2 - u} \{ J_t(0, x_2 - y^2) + J_t(l_0 - y^2, y^2) \}, \max_{x_2 - u \leq y^2 \leq l_0} \{ J_t(0, u) + J_t(l_0 - y^2, y^2) \} \right\}
\]

\[
= \max \{ J_t(0, u) + J_t(0, y^2) \}
\]

\[
= J_t(0, x_2) + J_t(l_0, 0).
\]

Case 2(iii): If $x_1 \leq l_0$ and $l_0 - x_1 \geq x_2 - u \geq 0$

\[
K_2 - (c - s)x_2 = \max \left\{ \max_{0 \leq y^2 \leq x_2 - u} \{ J_t(0, x_2 - y^2) + J_t(x_1, y^2) \}, \max_{0 \leq y^2 \leq l_0 - x_1} \{ J_t(0, u) + J_t(x_1, y^2) \} \right\}
\]

\[
= \max \left\{ \max_{0 \leq y^2 \leq x_2 - u} \{ J_t(0, x_2 - y^2) + J_t(x_1, y^2) \}, \max_{x_2 - u \leq y^2 \leq l_0 - x_1} \{ J_t(0, u) + J_t(x_1, y^2) \} \right\}
\]

When $u \leq l_0$, we can show that $O(x_1) \leq x_0 - x_1$. Thus if $x_2 - u \leq O(x_1)$, $K_2(x_1, x_2) = (c - s)x_2 + J_t(0, u) + J_t(x_1, O(x_1))$.

Case 2(iv): If $x_1 \leq l_0$ and $l_0 - x_1 \leq x_2 - u \leq l_0$

\[
K_2 - (c - s)x_2 = \max \left\{ \max_{l_0 - x_1 \leq y^2 \leq x_2 - u} \{ J_t(0, x_2 - y^2) + J_t(l_0 - y^2, y^2) \}, \max_{0 \leq y^2 \leq l_0 - x_1} \{ J_t(0, x_2 - y^2) + J_t(x_1, y^2) \} \right\}
\]

\[
= \max \left\{ \max_{l_0 - x_1 \leq y^2 \leq x_2 - u} \{ J_t(0, x_2 - y^2) + J_t(l_0 - y^2, y^2) \}, \max_{x_2 - u \leq y^2 \leq l_0} \{ J_t(0, x_2 - y^2) + J_t(x_1, y^2) \} \right\}
\]

\[
= \max \{ J_t(0, u) + J_t(l_0 - y^2, y^2) \}
\]

Case 2(v): If $x_1 \leq l_0$ and $x_2 - u \geq l_0$

\[
K_2 - (c - s)x_2 = \max \left\{ \max_{l_0 \leq y^2 \leq x_2 - u} \{ J_t(0, x_2 - y^2) + J_t(l_0 - y^2, y^2) \}, \max_{0 \leq y^2 \leq l_0 - x_1} \{ J_t(0, x_2 - y^2) + J_t(x_1, y^2) \} \right\}
\]

\[
= \max \left\{ \max_{l_0 \leq y^2 \leq x_2 - u} \{ J_t(0, x_2 - y^2) + J_t(l_0 - y^2, y^2) \}, \max_{x_2 - u \leq y^2 \leq l_0} \{ J_t(0, x_2 - y^2) + J_t(x_1, y^2) \} \right\}
\]

\[
= \max \{ J_t(0, x_2 - y^2) + J_t(l_0, 0) \}.
\]

For Cases 2(iii),(iv),(v), we know that if $x_2 - u \geq O(x_1)$, for $0 \leq y^2 \leq x_2 - u$,

\[
\frac{\partial (J_t(0, x_2 - y^2) + J_t(x_1, y^2))}{\partial y^2} = \frac{(p - \alpha v)\Phi(x_2 - y^2) - (p + \theta)\Phi(x_1 + y^2) + (\theta + \alpha v)\Phi(y^2)}{2} + (p - c) - (p + \theta)\Phi(x_1 + y^2) + (\theta + \alpha v)\Phi(y^2),
\]
which is greater than zero for \( y^2 \leq O(x_1) \), and hence \( \bar{y}^2(x_1, x_2|K_2) > 0 \) if \( x_1 \leq l \). If \( x_1 \geq u \) and \( x_1 \geq x_2 \),

\[
\frac{\partial \{J_t(0, x_2 - y^2) + J_t(x_1, y^2)\}}{\partial y^2} = (p - \alpha \nu)\Phi(x_2 - y^2) - (p + \theta)\Phi(x_1 + y^2) + (\theta + \alpha \nu)\Phi(y^2)
\]

\[
\leq (p - \alpha \nu)\Phi(x_2) - (p + \theta)\Phi(x_1) + (\theta + \alpha \nu)\Phi(x_2 - u)
\]

\[
\leq (\theta + \alpha \nu)(\Phi(x_2 - u) - \Phi(x_1))
\]

\[
\leq 0,
\]

and

\[
\frac{\partial \{J_t(0, x_2 - y^2) + J_t(l_0 - y^2, y^2)\}}{\partial y^2} = -(p - s) + (p - \alpha \nu)\Phi(x_2 - y^2) + (\theta + \alpha \nu)\Phi(y^2)
\]

\[
\leq -(p - s) + (p - \alpha \nu)\Phi(x_2 + x_1 - l_0) + (\theta + \alpha \nu)\Phi(x_2 - u)
\]

\[
\leq -(p - s) + (p + \theta)\Phi(x_2)
\]

\[
\leq 0,
\]

and hence \( \bar{y}^2(x_1, x_2|K_2) = 0 \).

From the above discussion, if we let

\[
C(x_1) = \begin{cases} 
0 & \text{if } x_1 \leq u \\
x_1 & \text{if } u \leq x_1 \leq l_0 \\
l_0 & \text{if } x_1 \geq l_0
\end{cases}
\]

and \( B(x_1) = \sup\{x_2 \geq 0 : \bar{y}^2(x_1, x_2|K_2) = 0\} \). Then, all of the results in the theorems hold.

**Part II:** For the optimization problem \( K_1 \), it is easy to see that if \( x_1 \geq l_0 \) and \( x_2 \leq u \), then \( K_1(x_1, x_2) = (c - s)x_2 + J_t(0, u) + J_t(l_0, 0) \). Hence \( \bar{z}_1^2(x_1, x_2|K_1) = 0 \) for any \( x_1 \) and \( x_2 \).

For the optimization \( K_2 \), we consider the following cases.

Case 1. If \( x_2 \leq 2u \), then \( K_2(x_1, x_2) = (c - s)x_2 + J_t(0, u) + J_t(0, u) \). Since \( \bar{z}_1^2(x_1, x_2|K_2) \) is decreasing in \( x_2 \), \( \bar{z}_1^2(x_1, x_2|K_2) = 0 \) for all \( x_2 \).

Case 2. If \( 2u \leq x_2 \leq 2u_0 \), then \( K_2(x_1, x_2) = (c - s)x_2 + J_t(0, x_2/2) + J_t(0, x_2/2) \).

From the above discussion, if we define \( B(x_1) = C(x_1) = 0 \), then all of the results in the theorem hold.

**Part III:** Since \( J_t(l_0 - y, y) \) is convex in \( y \), we let \( \tilde{l} = \sup\{y \in [0, l_0] : J_t(l_0 - y, y) \geq J_t(0, u)\} \).

It is easy to see that \( s - c + (\theta + \alpha \nu)\Phi(\tilde{l}) \leq 0 \). We first derive the optimal policy for the optimization problem \( K_1 \).

Case 1(i): If \( x_1 \leq l_0 \) and \( x_2 \leq x_1 \),

\[
K_1 - (c - s)x_2 = \max\{\max_{y^1 \geq x_1} \{J_t(0, y^1) + J_t(x_1, 0)\}, \max_{x_2 \leq y^1 \leq x_1} \{J_t(\frac{x_1 - y^1}{y^1}, y^1) + J_t(\frac{x_1 + y^1}{y^1}, 0)\}\}.
\]
In this case, if \( x_1 \leq l, K_1(x_1, x_2) = (c - s)x_2 + J_t(0, x_2) + J_t(x_2, 0). \) If \( x_1 \geq l, \)

\[
K_1(x_1, x_2) = \max \{ J_t(0, u) + J_t(x_1, 0) \}, \max_{x_2 \leq y_1 \leq x_1} \{ J_t \left( \frac{x_1 - y_1}{2}, y_1 \right) + J_t \left( \frac{x_1 + y_1}{2}, 0 \right) \}.
\]

Case 1(ii): If \( x_1 \leq l_0 \) and \( x_2 > x_1, \)

\[
K_1 - (c - s)x_2 = \max_{y_1 \geq x_2} \{ J_t(0, y_1) + J_t(x_1, 0) \}
= J_t(0, \max \{ x_2, u \}) + J_t(x_1, 0).
\]

Case 2(i): If \( x_1 \geq l_0 \) and \( x_2 \leq 2l_0 - x_1, \)

\[
K_1 - (c - s)x_2 = \max \{ J_t(0, l_0) + J_t(l_0, 0) \}, \max_{y_1 \geq x_2} \{ J_t(x_1 - l_0, y_1) + J_t(l_0, 0) \},
= \max_{x_2 \leq y_1 \leq 2l_0 - x_1} \{ J_t \left( \frac{x_1 - y_1}{2}, y_1 \right) + J_t \left( \frac{x_1 + y_1}{2}, 0 \right) \}.
\]

Case 2(ii): If \( x_1 \geq l_0 \) and \( 2l_0 - x_1 \leq x_2 \leq l_0, \)

\[
K_1 - (c - s)x_2 = \max \{ \max_{x_2 \leq y_1 \leq l_0} \{ J_t(x_1 - l_0, y_1) + J_t(l_0, 0) \}, \max_{y_1 \geq x_2} \{ J_t(0, y_1) + J_t(l_0, 0) \} \}
= J_t(l_0, 0) + \max \{ J_t(0, u), J_t(l_0 - x_2, x_2) \}.
\]

Thus, if \( x_2 \geq \tilde{l}, K_1 = (c - s)x_2 + J_t(0, u) + J_t(l_0, 0). \) If \( x_2 \leq \tilde{l}, K_1 = (c - s)x_2 + J_t(l_0 - x_2, x_2) + J_t(l_0, 0). \)

Case 2(iii): If \( x_1 \geq l_0 \) and \( x_2 \geq l_0, \)

\[
K_1 - (c - s)x_2 = \max_{y_1 \geq x_2} \{ J_t(0, y_1) + J_t(l_0, 0) \}, \max_{y_1 \geq x_2} \{ J_t(x_1 - l_0, y_1) + J_t(l_0, 0) \}
= J_t(0, \max \{ u, x_2 \}) + J_t(l_0, 0).
\]

Similarly, we can characterize the optimal policy for the optimization problem \( K_2, \) Notice that

\[
\frac{\partial J_t(x_1, y)}{\partial y}_{y = l_0 - x_1} = (\theta + \alpha v) (\Phi(l_0 - x_1) - \frac{c - s}{\theta + \alpha v}).
\]

Let \( \bar{u} = \sup \{ x_1 : \Phi(l_0 - x_1) \geq \frac{c - s}{\theta + \alpha v} \}. \) Thus, \( \max_{0 \leq y^2 \leq l_0 - x_1} J_t(x_1, y^2) = J_t(x_1, l_0 - x_1) \) for \( x_1 \leq \bar{u} \) and \( \max_{0 \leq y^2 \leq l_0 - x_1} J_t(x_1, y^2) = J_t(x_1, O(x_1)) \) otherwise. Let \( u' = \inf \{ x_1 \geq \bar{u} : J_t(x_1, O(x_1)) \geq J_t(0, u) \}. \)

Case 1(i): If \( x_2 \leq u \) and \( x_1 \geq l_0, \)

\[
K_2 - (c - s)x_2 = \max_{0 \leq z_1^2 \leq l_0, y^2 \geq 0} \{ J_t(0, u) + J_t(z^2_1, y^2) \}
= \max_{0 \leq y^2 \leq l_0} \{ J_t(0, u) + J_t(l_0 - y^2, y^2) \}, \max \{ J_t(0, u) + J_t(0, y^2) \}
= J_t(0, u) + J_t(l_0, 0).
\]
Case 1(ii): If $x_2 \leq u$ and $x_1 \leq l_0$, 

$$K_2 - (c - s)x_2 = \max_{0 \leq z_1^2 \leq x_1, y^2 \geq 0} \left\{ J_t(0, u) + J_t(z_1^2, y^2) \right\}$$

$$= \max_{0 \leq y^2 \leq l_0 - x_1} \left\{ J_t(0, u) + J_t(l_0 - y^2, y^2) \right\}, \quad \max_{0 \leq y^2 \leq l_0 - x_1} \left\{ J_t(0, u) + J_t(x_1, y^2) \right\}$$

$$\max_{y^2 \geq l_0} \left\{ J_t(0, u) + J_t(0, y^2) \right\}.$$ 

In this case, if $x_1 \leq u'$, $K_2(x_1, x_2) = (c - s)x_2 + J_t(0, u) + J_t(0, u)$; and if $x_1 \geq u'$, $K_2(x_1, x_2) = (c - s)x_2 + J_t(0, u) + J_t(x_1, O(x_1)).$

Case 2(i): If $x_1 \geq l_0$ and $x_2 - u \geq l_0$, 

$$K_2 - (c - s)x_2 = \max_{0 \leq z_1^2 \leq x_1, 0 \leq y^2 \leq x_2 - u} \left\{ J_t(0, x_2 - y^2) + J_t(z_1^2, y^2) \right\}, \quad \max_{0 \leq z_1^2 \leq x_1, y^2 \geq x_2 - u} \left\{ J_t(0, u) + J_t(z_1^2, y^2) \right\}$$

$$= \max_{0 \leq y^2 \leq l_0} \left\{ J_t(0, x_2 - y^2) + J_t(l_0 - y^2, y^2) \right\}, \quad \max_{0 \leq y^2 \leq l_0} \left\{ J_t(0, x_2 - y^2) + J_t(0, y^2) \right\},$$

$$\max_{y^2 \geq l_0} \left\{ J_t(0, u) + J_t(0, y^2) \right\}.$$ 

Case 2(ii): If $x_1 \geq l_0$ and $u \leq x_2 \leq u + l_0$

$$K_2 - (c - s)x_2 = \max_{0 \leq z_1^2 \leq x_1, 0 \leq y^2 \leq x_2 - u} \left\{ J_t(0, x_2 - y^2) + J_t(z_1^2, y^2) \right\}, \quad \max_{0 \leq z_1^2 \leq x_1, y^2 \geq x_2 - u} \left\{ J_t(0, u) + J_t(z_1^2, y^2) \right\}$$

$$= \max_{0 \leq y^2 \leq x_2 - u} \left\{ J_t(0, x_2 - y^2) + J_t(l_0 - y^2, y^2) \right\}, \quad \max_{x_2 - u \leq y^2 \leq l_0} \left\{ J_t(0, u) + J_t(l_0 - y^2, y^2) \right\},$$

$$\max_{y^2 \geq l_0} \left\{ J_t(0, u) + J_t(0, y^2) \right\}.$$ 

For case 2(i) and (ii), we can show that if $x_2 \geq u + \bar{t}$,

$$K_2 - (c - s)x_2 = \max_{0 \leq y^2 \leq l} \left\{ J_t(0, x_2 - y^2) + J_t(l_0 - y^2, y^2) \right\}, \quad 2J_t(0, \max\{u, x_2/2\});$$

and if $x_2 \leq u + \bar{t}$,

$$K_2 - (c - s)x_2 = \max_{0 \leq y^2 \leq x_2 - u} \left\{ J_t(0, x_2 - y^2) + J_t(l_0 - y^2, y^2) \right\}.$$ 

Case 2(iii): If $x_1 \leq l_0$ and $l_0 - x_1 \geq x_2 - u \geq 0$,

$$K_2 - (c - s)x_2 = \max_{0 \leq y^2 \leq x_2 - u} \left\{ J_t(0, x_2 - y^2) + J_t(x_1, y^2) \right\}, \quad \max_{0 \leq y^2 \leq x_2 - u} \left\{ J_t(0, u) + J_t(l_0 - y^2, y^2) \right\},$$

$$\max_{x_2 - u \leq y^2 \leq l_0} \left\{ J_t(0, u) + J_t(l_0 - y^2, y^2) \right\}, \quad \max_{x_2 - u \leq y^2 \leq l_0} \left\{ J_t(0, u) + J_t(x_1, y^2) \right\}.$$
In this case, if \( x_1 \leq u' \), \( K_2(x_1, x_2) = (c-s)x_2 + J_l(0, u) + J_l(0, u) \). If \( x_1 \geq u' \) and \( x_2 - u \leq O(x_1) \), \( K_2(x_1, x_2) = (c-s)x_2 + J_l(0, u) + J_l(0, x_1, O(x_1)) \). And if \( x_1 \geq u' \) and \( x_2 - u \geq O(x_1) \),

\[
K_2 - (c-s)x_2 = \max\{ \max_{0 \leq y^2 \leq x_2-u} \{J_l(0, x_2 - y^2) + J_l(x_1, y^2)\}, J_l(0, u) + J_l(0, u) \}.
\]

**Case 2(iv):** If \( x_1 \leq l_0 \) and \( l_0 - x_1 \leq x_2 - u \leq l_0 \),

\[
K_2 - (c-s)x_2 = \max\{ \max_{l_0-x_1 \leq y^2 \leq x_2-u} \{J_l(0, x_2 - y^2) + J_l(l_0 - y^2, y^2)\}, \max_{0 \leq y^2 \leq l_0-x_1} \{J_l(0, x_2 - y^2) + J_l(x_1, y^2)\}, \max_{x_2-u \leq y^2 \leq l_0} \{J_l(0, u) + J_l(l_0 - y^2, y^2)\} \}.
\]

**Case 2(v):** If \( x_1 \leq l_0 \) and \( x_2 - u \geq l_0 \),

\[
K_2 - (c-s)x_2 = \max\{ \max_{l_0-x_1 \leq y^2 \leq l_0-u} \{J_l(0, x_2 - y^2) + J_l(l_0 - y^2, y^2)\}, \max_{0 \leq y^2 \leq l_0-x_1} \{J_l(0, x_2 - y^2) + J_l(x_1, y^2)\}, \max_{l_0 \leq y^2 \leq x_2-u} \{J_l(0, x_2 - y^2) + J_l(0, y^2)\} \}.
\]

For Cases 2(iv) and (v), we can show that if \( x_1 \leq u' \), we have \( K_2-(c-s)x_2 = 2J_l(0, \max\{x_2, u\}/2) \); and if \( x_1 \geq u' \), we have

\[
K_2 - (c-s)x_2 = \max\{ \max_{l_0-x_1 \leq y^2 \leq l_0-u} \{J_l(0, x_2 - y^2) + J_l(l_0 - y^2, y^2)\}, \max_{O(x_1) \leq y^2 \leq l_0-x_1} \{J_l(0, x_2 - y^2) + J_l(x_1, y^2)\}, 2J_l(0, \max\{x_2, u\}/2) \}.
\]

Let \( C(x_1) = \inf\{x_2 \geq 0 : \tilde{z}_1(x_1, x_2|K_1) = 0\} \) and \( B(x_1) = \sup\{x_2 \geq 0 : \tilde{y}^2(x_1, x_2|K_2) = 0\} \), then all the results hold. In addition, from the above discussion, we know that if \( u' \leq l \), \( C(x_1) \) must be smaller than

\[
\tilde{C}(x_1) = \begin{cases} 
0 & \text{if } x_1 \leq l; \\
\tilde{l} & \text{if } x_1 \geq l
\end{cases}
\]

and \( B(x_1) \) must be greater than

\[
\tilde{B}(x_1) = \begin{cases} 
0 & \text{if } x_1 \leq l; \\
u & \text{if } x_1 \geq l.
\end{cases}
\]

If \( u' \geq l \), \( C(x_1) \) must be smaller than

\[
\tilde{C}(x_1) = \begin{cases} 
0 & \text{if } x_1 \leq u'; \\
\tilde{l} & \text{if } x_1 \geq u'
\end{cases}
\]

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and $B(x_1)$ must be greater than

$$
\hat{B}(x_1) = \begin{cases} 
0 & \text{if } x_1 \leq u'; \\
 u & \text{if } x_1 \geq u'.
\end{cases}
$$

\[ \Box \]

**Proof of Theorem 9**: The proof consists of two steps.

Step 1. We first prove that if $\bar{z}_1^1 + \bar{y}^1 > \bar{y}^2$, either $\bar{z}_1^1 = 0$ or $\bar{y}^2 = 0$. The proof is by contradiction. Suppose for all optimal policies that satisfy $\bar{z}_1^1 + \bar{y}^1 > \bar{y}^2$, we have $\bar{z}_1^1 > 0$ and $\bar{y}^2 > 0$. Let $\delta = \min(\bar{z}_1^1, \bar{y}^2)$ and we construct a new policy as follows:

$$
\begin{align*}
\bar{z}_1^1 &= \bar{z}_1^1 - \delta, & \bar{z}_2^1 &= \bar{z}_2^1 + \min(\delta, \bar{z}_2^2), & y_1^1 &= \bar{y}^1 + \delta, \\
\bar{z}_2^1 &= \bar{z}_1^1 + \delta, & \bar{z}_2^2 &= \bar{z}_2^2 - \min(\delta, \bar{z}_2^2), & y_2^1 &= \bar{y}^2 - \delta,
\end{align*}
$$

It is not difficult to show that the new policy is still feasible and either $\bar{z}_1^1$ or $\bar{y}^2$ is zero and $\bar{z}_1^1, \bar{z}_2^1, y_1^1, \bar{z}_2^2, \bar{y}^2$ are all nonnegative. The objective function $J_t$ can be written as

$$
J_t(z_t^1, y_t^1) = -sz_t^1 - cy^t + p\min(D^i, z_t^i + y^i) - \theta\min(z_t^1 + y^i - D^i)^+ + (\theta + \alpha v)\min(y^i - D^i)^+.
$$

The expected profit under the new policy minus that under the optimal policy is

$$
(\theta + \alpha v)[E(\bar{y}^1 + \delta - D^1)^+ + E(\bar{y}^2 - \delta - D^2)^+ - E(\bar{y}^1 - D^1)^+ - E(\bar{y}^2 - D^2)^+].
$$

We have

$$
E(\bar{y}^1 + \delta - D^1)^+ - E(\bar{y}^1 - D^1)^+ \geq E(\bar{y}^2 - D^1)^+ - E(\bar{y}^2 - \delta - D^1)^+, \\
\geq E(\bar{y}^2 - D^2)^+ - E(\bar{y}^2 - \delta - D^2)^+.
$$

Here the first inequality holds because the function $E(x - D^1)^+$ is a convex function in $x$ and $\bar{y}^1 + \delta \geq \bar{y}^2$. The second is true because $(x - y)^+$ is submodular in $(x, y)$, and $D^2$ is larger than $D^1$ stochastically. Therefore, the new policy achieves a higher profit than the optimal policy, which is a contradiction.

Step 2. We now prove that if $\bar{z}_1^1 + \bar{y}_1 > \bar{y}^2$, then $\bar{z}_2^2 > 0$.

Step 2(i). We first prove that if $\bar{z}_1^1 > 0$, then $\bar{z}_1^1 + \bar{y}^1 \leq \bar{z}_2^2 + \bar{y}^2$. The proof is by contradiction. Suppose in all optimal solutions, $\bar{z}_1^1 + \bar{y}_1 > \bar{z}_2^2 + \bar{y}^2$. Consider a new policy with $\bar{z}_1^1 = \bar{z}_1^1 - \delta$, $\bar{z}_2^2 = \bar{z}_2^2 + \delta$, $y^k = \bar{y}^k$ and $z^k = \bar{z}_2^k$ for $k = 1, 2$. The new policy is feasible as long as $0 \leq \delta \leq \bar{z}_1^1$. 

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It is easy to see that the value function under new policy is given by
\[
\sum_{i=1}^{2} s(x_i - z_i^1 - z_i^2) - \sum_{i=1}^{2} c(y_i - z_i^1) \\
+ p(\bar{z}_1 + \bar{y}_1 + \bar{z}_2 + \bar{y}_2) \\
- (p + \theta)[E(\bar{z}_1 + \bar{y}_1 - \delta - D^1)^+ + E(\bar{z}_2 + \bar{y}_2 + \delta - D^2)^+] \\
+ (\theta + \alpha v)[E(\bar{y}_1 - D^1)^+ + E(\bar{y}_2 - D^2)^+].
\]

Its first order derivative with respect to \( \delta \) is
\[
(p + \theta)[\Phi_1(\bar{z}_1 + \bar{y}_1 - \delta) - \Phi_2(\bar{z}_2 + \bar{y}_2 + \delta)].
\]

Because \( D^2 \geq_{st} D^1 \), \( \Phi_2(x) \leq \Phi_1(x) \). The above derivative is greater than zero as long as \( \bar{z}_1 + \bar{y}_1 - \delta \geq \bar{z}_2 + \bar{y}_2 + \delta \). This means that we can find a new policy that is better than the optimal policy and either \( \bar{z}_1 = 0 \) or \( \bar{z}_1 + \bar{y}_1 = \bar{z}_2 + \bar{y}_2 \), which is a contradiction.

Step 2(ii). We shall prove that if \( \bar{z}_1^2 = 0 \), then \( \bar{y}_1 \leq \bar{y}_2 \). The proof is by contradiction. Suppose in all optimal policies, \( \bar{y}_1 > \bar{y}_2 \). We construct a new policy:
\[
\bar{z}_1 = \bar{z}_1, \quad \bar{y}_1 = \bar{y}_1 - \delta, \quad \bar{z}_2 = \bar{z}_2 - \min\{\delta, \bar{z}_1\}, \\
\bar{z}_1 = \bar{z}_1, \quad \bar{y}_2 = \bar{y}_2 + \delta, \quad \bar{z}_2 = \bar{z}_2 + \min\{\delta, \bar{z}_1\}.
\]

It is easy to see that the new policy is feasible as long as \( 0 \leq \delta \leq \bar{y}_1 \). The value function under new policy is given by
\[
\sum_{i=1}^{2} s(\sum_{i=1}^{2} (x_i - \bar{z}_i^1 - \bar{z}_i^2)) - \sum_{i=1}^{2} c(\sum_{i=1}^{2} (y_i - \bar{z}_i^1)) \\
+ p(\bar{z}_1 + \bar{y}_1 + \bar{z}_2 + \bar{y}_2) \\
- (p + \theta)[E(\bar{z}_1^2 + \bar{y}_1 - \delta - D^1)^+ + E(\bar{z}_2 + \bar{y}_2 + \delta - D^2)^+] \\
+ (\theta + \alpha v)[E(\bar{y}_1 - D^1)^+ + E(\bar{y}_2 - D^2)^+].
\]

Its first order derivative with respect to \( \delta \) is
\[
-(p + \theta)[\Phi_2(\bar{y}_2 + \delta) - \Phi_1(\bar{z}_1 + \bar{y}_1 - \delta)) + (\theta + \alpha v)(\Phi_2(\bar{y}_2 + \delta) - \Phi_1(\bar{y}_1 - \delta)) \\
\geq -(p - \alpha v)(\Phi_2(\bar{y}_2 + \delta) - \Phi_1(\bar{y}_1 - \delta)) \\
\geq 0.
\]

The last inequality holds if \( 0 \leq \delta \leq (\bar{y}_1 - \bar{y}_2^2)/2 \). Therefore, we can find a new policy that is better than the optimal policy and \( \bar{y}_1 = \bar{y}_2 \), which is a contradiction.

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Step 2(iii). We have already proved that if $\bar{z}_2^2 = 0$, then $\bar{y}_1^1 \leq \bar{y}_2^2$. Now if we further have $\bar{z}_1^1 = 0$, then $\bar{z}_1^1 + \bar{y}_1^1 \leq \bar{y}_2^2$. Otherwise if $\bar{z}_1^1 > 0$, then from Step 2(i), we have $\bar{z}_1^1 + \bar{y}_1^1 \leq \bar{z}_2^2 + \bar{y}_2^2 = \bar{y}_2^2$. Together, this means that if $\bar{z}_2^2 = 0$, we have $\bar{z}_1^1 + \bar{y}_1^1 \leq \bar{y}_2^2$. Equivalently, this means if $\bar{z}_1^1 + \bar{y}_1^1 > \bar{y}_2^2$, then $\bar{z}_1^2 > 0$.

References


