A Recursive Dynamic Agency Model  
and a Semi-Linear First-Best Contract  

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Abstract  

A recursive dynamic agency model is developed for situations where the state of nature follows a Markov process. The repeated agency model is a special case. It is found that the optimal effort depends not only on current performance but also on past performances, and the disparity between current and past performances is a crucial determinant in the optimal contract. In a special case when both the principal and the agent are risk neutral, the first best is achieved by a semi-linear contract. In another special case of a repeated version, the first best is again achieved when the discount rate converges to zero. For the general model, a computing algorithm is developed, which can be implemented in MathCAD to find the solution numerically.
1. Introduction

Dynamic agency models have recently shown their power in explaining many economic phenomena and are increasingly been used in many fields such as industrial organization, labor economics, health economics, insurance, and foreign trade; see, for example, Laffont (1995), Green (1987) and Wang (1995, 1997).

However, existing infinite-period dynamic agency models are almost exclusively ‘repeated’ agency models in which the underlying state of nature is an i.i.d process. This type of models were first introduced by Spear–Srivastava (1987) and Green (1987), and they have been subject to many refinements and further developments in recent years; see, for example, Abreu–Pearce–Stacchetti (1990), Thomas–Worrall (1990), Atkeson–Lucas (1992) and Wang (1995, 1997). In these models, although the agents do have concern for future welfare, the nature of uncertainty from period to period simply repeats itself and the evolving nature of a typical dynamic model is missing. For example, the optimal effort level stays the same throughout all contracting periods; it does not change according the state of nature.

This paper develops a dynamic agency model that allows the underlying state of nature to be a Markov process. It includes the repeated agency model as a special case. By this, we have an infinite-period dynamic agency model that is suitable for a wide variety of agency problems. We call this type of dynamic agency models recursive agency models, in contrast with repeated agency models.

The word ‘repeated’ comes from repeated games. As we know, the game of Prisoners Dilemma may yield the efficient solution if it is played repeatedly and has small discount of future payoffs. Similarly, although the static principal-agent model typically fails to reach the first best, it has been shown by Thomas–Worrall (1990), among others, that in a repeated agency model, the efficiency of the optimal contract is improved and it can even reach the first best if there is no discounting of the future. We show that this result holds for a repeated version of our model.

Our main interest in a recursive agency model is in the dynamic nature of the optimal contract. As we will show, the optimal contract depends not only on current performance but also on past performances. Instead of contracting solely on current performance, the principal in a recursive dynamic setting can also use past information as a basis for contracting. In this way, the agent need not be awarded highly for a good performance if the economy was in a good state during the last period. The reason is that, with a serially correlated sequence of outputs, if the economy was in a good state during the last period, a good performance is likely due to a good state of the economy rather than a big effort. It is also found that the disparity between current and past performances is a crucial determinant in the optimal contract. The principal not only uses outputs to reward and punish, but also uses them to speculate on the effort inputs from the agent. In other words, the principal relies heavily on relative performance to improve efficiency in a dynamic contract.
In repeated agency models, the ‘state of nature’ is the best utility foregone for the agent. However, the best utility foregone is unobservable, which makes it difficult to test a theory empirically. Output and profit processes on the other hand are observable. In this paper, we will assume that an agent’s utility value is not verifiable. Since outputs and profits are highly correlated over time in reality, especially for monthly data, by allowing the state of nature in an agency model to be Markovian, we can use an output or profit process to be the state of nature and the results will be testable by real-world data.

The paper is organized as follows. We will present our model in Section 2 along with the basic static agency model and a repeated agency model. Two alternative setups of our model are also discussed. A basic analysis of the solution from our model is presented in Section 3. Section 4 presents an interesting special case under risk neutrality, for which a semi-linear first-best contract is found. Just as the basic static agency model does, there is generally no closed-form solution for the general case. Section 5 thus presents a computing algorithm and then uses it to solve for a numerical example. Some interesting observations from our computing solution are discussed. Section 6 concludes the paper with a few remarks. An appendix is also attached for the proofs of Propositions 1–5.

The Model

1.1. Three Agency Models

Consider a principal-agent relationship. A principal hires an agent for producing a product. The output \( x \in \mathbb{Y} \) is random, where \( \mathbb{Y} \) is an arbitrary Lebesgue-measurable set, \( \mathbb{Y} \subset \mathbb{R} \). For the static agency model, given the agent’s effort \( a \in \mathbb{A} \), where \( \mathbb{A} \) is also an arbitrary Lebesgue-measurable set, \( \mathbb{A} \subset \mathbb{R} \), the output follows a process that can be described by a density function \( f(x|a) \), defined for \( x \in \mathbb{Y} \). The principal’s preferences are described by a utility function \( v(x) \) of payoff \( x \), and the agent’s preferences are described by a utility function \( u(x,a) \) of effort \( a \) and payoff \( x \). We assume

\[
\begin{align*}
    v' > 0, & \quad v'' \leq 0, \quad u_x > 0, \quad u_{xx} < 0, \quad u_a < 0.
\end{align*}
\]

The principal can observe and verify the output \( x \) but cannot verify the effort \( a \); she thus offers the agent a contract \( s(x) \) that is based on the output, where the contract is drawn from the following contract space

\[
S_0 = \{ s : \mathbb{Y} \rightarrow \mathbb{R}_+ \mid s \text{ is Lebesgue-measurable} \}.
\]

Notice that we require \( s(x) \geq 0 \) and call it the limited liability condition. Finally, let \( \bar{U} \) be the agent’s best alternative utility foregone if the agent accepts a contract from the principal. The following is the basic static agency model for the agency problem:
That is, the principal maximizes her own utility subject to the incentive compatibility (IC) condition and the individual rationality (IR) condition. This model has been extensively discussed in the literature; see, for example, Mirrless (1974), Holmstrom (1979), Grossman–Hart (1983), Holmstrom–Milgrom (1987) and Kim–Wang (1998). The solution of this basic model is generally the second best due to moral hazard on the agent’s part.

To deal with the inefficiency of the second-best solution from (2.1), a repeated agency model is proposed. When output process \( \{y_t\}_{t=0}^\infty \) is i.i.d, Spear–Srivastava (1987) propose an infinite-period version of the agency problem:

\[
V(w) = \max_{a(w), U(\cdot, w), s(\cdot, w)} \int \{v[x - s(x, w)] + \alpha V[U(x, w)]\} f[x|a(w)] dx \\
\text{s.t.} \quad a(w) \in \arg \max_{a' \in A} \int \{u[s(x, a'), a'] + \beta U(x, a')\} f(x|a) dx \\
\int \{u[s(x, a), a] + \beta U(x, a)\} f[x|a(w)] dx \geq w,
\]

for all \( w \in \mathbb{W} \), where \( w \) is the best alternative utility foregone if the agent accepts a contract from the principal. The solution will give four functions \( a(\cdot), U(\cdot, \cdot), s(\cdot, \cdot) \) and \( V(\cdot) \). There is also a large and recent literature on this model; see, for example, Rogerson (1985a), Abreu–Pearce–Stacchetti (1990), Atkeson–Lucas (1992), and Wang (1995, 1997). In particular, Thomas–Worrall (1990) show that in a repeated agency model, the efficiency of the optimal contract is improved and it can even reach the first best if there is no discounting for the future.

It is natural to extend a dynamic agency model to a model that allows the state of nature to follow a Markov process. Then, what is the proper formulation of the agency problem if output process \( \{y_t\}_{t=0}^\infty \) is Markovian, i.e. \( y_{t+1} \sim f(\cdot | y_t, a_t) \), for a given density function \( f \)? Given a utility function \( U(y) \) representing the best alternative foregone in the market for each output \( y \), we will show that the following is a proper formulation of the agency

\[
1 \text{Since the second constraint in problem (2.2) holds for any } w, \text{ it must also hold for } U(w, y). \text{ Thus, we must have}
\]

\[
U(w, y) = \int \{u(sU(w, y), y'), a[U(w, y)]\} f(y'|a[U(w, y)]) dy',
\]

for all \( w \in \mathbb{W} \) and \( y \in Y \). Spear-Srivastava (1987) listed this condition as one of the three constraints, but it is actually redundant.
problem:

\[
V(y) = \max_{a(y), U(\cdot), s(\cdot, y)} \int \left\{ v[x - s(x, y)] + \alpha V(x) \right\} f[x|y, a(y)] dx
\]

s.t. 
\[
a(y) \in \arg \max_{a \in k} \int \left\{ u[s(x, y), a] + \beta U(x) \right\} f[x|y, a] dx,
\]

\[
\int \left\{ u[s(x, y), a(y)] + \beta U(x) \right\} f[x|y, a(y)] dx = U(y),
\]

for all \( y \in \mathbb{Y} \). It is a recursive agency model. The solution is a tuple of four functions \((a, s, U, V)\). In this case, the solution will obviously depend on the initial utility \( w = U(y) \).

But, since \( U(\cdot) \) is a given function, we do not need to write the contract’s dependence on \( U \) explicitly. That is, for the solution of (2.3), the dependence of the solution on \( U(\cdot) \) is implicit; instead, we will use \( y_t \) as the state of nature at \( t \) and the solution will be specified as functions of the state. This treatment is consistent with the repeated model, in which \( w \) is treated as the state; in fact, (2.2) is a special case of (2.3) with \( y = U^{-1}(w) \) for a given \( w \).

This is due to the difference between the solution of (2.2) and that of (2.3) — the solution of (2.2) is an open-loop solution while the solution of (2.3) is a closed-loop solution.

The solution for our model (2.3) is time-consistent, due to the recursive nature of our model.

**Remark 1.** In this paper, we limit to the first-order Markov process of the form \( y_{t+1} \sim f(\cdot|y_t, a_t) \). It is straightforward to extend it to any finite-order Markov process. That is, our recursive model applies to any stationary random output process that depends on a fixed and finite number of periods of history.

**Remark 2.** Recursive models cover a large variety of dynamic models. In macroeconomics, economists almost never go beyond recursive models for theoretical analysis when an infinite-period model is used. The well-known paper by Lucas (1978), together with Stokey–Lucas (1989), lays the foundation of recursive models for macroeconomics. There are two key advantages of recursive models. First, the solutions are time consistent. Second, they often provide tractable solutions with attractive characterizations. In contrast, non-recursive models generally rely on numerical solutions.

**Remark 3.** One problem for model (2.2) is that the principal uses \( U(x, w) \) to represent the market condition in the next period. But the market condition can differ from \( U(x, w) \). For example, in a steady state, the market may still produce \( w \), as opposed to \( U(x, w) \), in the next period. The market condition is exogenously determined; the endogenously determined \( U(x, w) \) cannot possibly reflect that. Such inconsistency does not exist in our model (2.3); the market condition is represented by an exogenous process \( \{y_t\}_{t=0}^{\infty} \).
1.2. Setup for the Recursive Model

We now formally setup our model. At time $t$ with a given output-action combination $(y_t, a_t)$, denote $y_{t+1} \sim f(\cdot | y_t, a_t)$ as to mean that $y_{t+1}$ is conditionally distributed according to the density function $f(\cdot | y_t, a_t)$. We call $\{y_t\}_{t=0}^\infty$ a first-order stationary Markov process, described by $f : \mathbb{Y}^2 \times \mathbb{A} \rightarrow \mathbb{R}_+$. We denote $\mathbb{N} = \{0, 1, 2, \ldots \}$ as the set of all positive integers.

Assumption 1. (Markov State Process). $y_{t+1} \sim f(\cdot | y_t, a_t)$ for all $y_t \in \mathbb{Y}$, $a_t \in \mathbb{A}$ and $t \in \mathbb{N}$.

We call $f(\cdot | y_t, a_t)$ the transition density function of the Markov process $\{y_t\}_{t=0}^\infty$ with controls $\{a_t\}_{t=0}^\infty$. Given a contract $\{s_t\}_{t=1}^\infty$ specifying a sequence of payments $s_t$ to the agent, the utility function of the principal is $v : \mathbb{R} \rightarrow \mathbb{R}$, denoting the utility at $t$ as $v_t = v(y_t - s_t)$. The utility function of the agent is $u : \mathbb{R}_+ \times \mathbb{A} \rightarrow \mathbb{R}$, denoting the utility at $t$ as $u_t = u(s_t, a_{t-1})$. This is consistent with the standard way, as for example, in Spear–Srivastava (1987). The action $a_t$ is aimed at the utility $u_{t+1}$ in the next period. In addition, the action $a_t$ is taken based on the realized state $y_t$. We will restrict the contract space to the following space:

$$
\mathcal{S} \equiv \{ s : \mathbb{Y}^2 \rightarrow \mathbb{R}_+ \mid s \text{ is Lebesgue-measurable} \}.
$$

That is, we only consider feedback contracts that depend only on the current state of nature and the immediate past performance: $s_t = s(y_t, y_{t-1})$. We conjecture that more general contracts would not Pareto-dominate over our feedback contracts. By restricting to this type of contracts, we can transform a general dynamic agency problem into the recursive problem in (2.3). In this case, as we show later, we must have $a_{t-1} = a(y_{t-1})$. Then, the action space is

$$
\mathcal{A} \equiv \{ a : \mathbb{Y} \rightarrow \mathbb{A} \mid a \text{ is Lebesgue-measurable} \}.
$$

In summary, the sequences of variables $\{a_t, s_t, u_t, f_t\}$ are defined as

$$
a_t = a(y_t), \quad s_{t+1} = s(y_{t+1}, y_t), \quad u_{t+1} = u(s_{t+1}, a_t), \quad f_{t+1} = f(y_{t+1} | y_t, a_t).
$$

The expectation operator $E^{a(\cdot)}_t$ is defined by the Markov process $\{y_t\}_{t=0}^\infty$ and the action function $a(\cdot)$, meaning that for $s \leq t$ and any Borel-measurable function $\psi : \mathbb{Y} \rightarrow \mathbb{R}$, we have

$$
E^{a(\cdot)}_s[\psi(y_{t+1})] = \int \cdots \int \left\{ \int \psi(y_{t+1}) f(y_{t+1} | y_t, a(y_t)) dy_{t+1} \right\} f(y_t | y_{t-1}, a(y_{t-1})) dy_t \cdots dy_s + 1.
$$

\footnote{For simplicity, we do not impose the condition $s_t \leq y_t$ for all $t \in \mathbb{N}$ in our presentation. However, the numerical example in Section 5.2 does satisfy this condition.}
where \( y_s \) is given. Similarly, the expectation operator \( E_t^{[a_t]} \) is defined by the Markov process \( \{y_t\}_{t=0}^\infty \) and the action series \( \{a_t\} \), meaning that for \( s \leq t \) and any Borel-measurable function \( \psi : \mathcal{Y} \to \mathbb{R} \), we have

\[
E_s^{[a_s]}[\psi(y_{t+1})] = \int \cdots \int \left\{ \int \psi(y_{t+1})f(y_{t+1}|y_t, a_t)dy_{t+1} \right\} f(y_t|y_{t-1}, a_{t-1})dy_t \cdots dy_{s+1}.
\]

Also, define

\[
E_s^y[\psi(x)] = \int \psi(x)f(x|y, a)dx.
\]

Denote the history as \( h^t \equiv (y_0, y_1, \ldots, y_t) \) and \( \mathcal{L}[\mathcal{Y}] \) as

\[
\mathcal{L}[\mathcal{Y}] \equiv \left\{ U : \mathcal{Y} \to \mathbb{R} \mid U \text{ is Lebesgue-measurable} \right\}.
\]

Furthermore, for convenience, we will use notation \( y = y_t \) and \( x = y_{t+1} \) when there is no confusion.

Consider a formal setting of the principal’s problem:

\[
V(y_0) = \max_{\{s(h^t), \{a(h^t)\}\}} E_0^{\left\{a(h^t)\right\}} \sum_{t=1}^\infty \alpha^{t-1}v[y_t - s(h^t)]
\]

s.t. \( y_{t+1} \sim f[\cdot | y_t, a(h^t)], \quad \forall t \geq 0, \)

\[
\{a(h^t)\} \in \arg \max_{\{a_t\} \in \mathcal{A}^\infty} E_0^{[a_t]} \sum_{t=1}^\infty \beta^{t-1}u[s(h^t), a_{t-1}] \quad \text{s.t.} \quad y_{t+1} \sim f(\cdot | y_t, a_t),
\]

\[
E_0^{\left\{a(h^t)\right\}} \sum_{t=1}^\infty \beta^{t-1}u[s(h^t), a(h^{t-1})] \geq \bar{U}(y_0), \quad \forall y_0 \in \mathcal{Y}.
\]

Since we only consider feedback contracts in \( \mathcal{S} \) with the form \( s_t = s(y_t, y_{t-1}) \), the problem can be rewritten as

\[
V(y_0) = \max_{s(\cdot) \in \mathcal{S}, \{a(h^t)\}} E_0^{\left\{a(h^t)\right\}} \sum_{t=1}^\infty \alpha^{t-1}v[y_t - s(y_t, y_{t-1})]
\]

s.t. \( y_{t+1} \sim f[\cdot | y_t, a(h^t)], \quad \forall t \geq 0, \)

\[
\{a(h^t)\} \in \arg \max_{\{a_t\} \in \mathcal{A}^\infty} E_0^{[a_t]} \sum_{t=1}^\infty \beta^{t-1}u[s(y_t, y_{t-1}), a_{t-1}] \quad \text{s.t.} \quad y_{t+1} \sim f(\cdot | y_t, a_t),
\]

\[
E_0^{\left\{a(h^t)\right\}} \sum_{t=1}^\infty \beta^{t-1}u[s(y_t, y_{t-1}), a(h^{t-1})] \geq \bar{U}(y_0), \quad \forall y_0 \in \mathcal{Y}.
\]

The first objective is to prove that (2.3) and (2.4) are the same. The result is stated in Proposition 1, and its proof is provided in the appendix.

**Proposition 1. (Recursiveness).** Given \( \bar{U} \in \mathcal{L}[\mathcal{Y}] \), any solution of (2.4) must be a solution of (2.3), where for (2.3), \( a \in \mathcal{A} \), \( s \in \mathcal{S} \) and \( V, U \in \mathcal{L}[\mathcal{Y}] \).
1.3. Alternative Setups

There are two alternative setups for (2.3). In the first alternative setup, the IR condition is satisfied only for the initial market constraint. In this case, given a utility level $\bar{U}$ of the best alternative opportunity at the start of the contract $t = 0$, the problem can be expressed as

$$V(y) = \max_{a(\cdot) \in A, \ U(\cdot) \in \mathcal{L}[Y], \ s(\cdot) \in S} \int \left\{ v[x - s(x, y)] + \alpha V(x) \right\} f[x|y, a(y)]dx$$

subject to

$$a(y) \in \arg \max_{a \in \mathcal{A}} \int \left\{ u[s(x, y), a] + \beta U(x) \right\} f(x|y, a)dx,$$

$$\int \left\{ u[s(x, y), a(y)] + \beta U(x) \right\} f[x|y, a(y)]dx = U(y),$$

$$U(y_0) \geq \bar{U}.$$  

For this case, the contract ensures acceptance at $t = 0$, but it may need enforcement in the future. If $U(y_t) < \bar{U}$ at a future date $t$, the agent would want to break the contract.

In the second alternative setup, we ignore outside competition altogether and consider a simpler version:

$$V(y) = \max_{a(\cdot) \in A, \ U(\cdot) \in \mathcal{L}[Y], \ s(\cdot) \in S} \int \left\{ v[x - s(x, y)] + \alpha V(x) \right\} f[x|y, a(y)]dx$$

subject to

$$a(y) \in \arg \max_{a \in \mathcal{A}} \int \left\{ u[s(x, y), a] + \beta U(x) \right\} f(x|y, a)dx,$$

$$\int \left\{ u[s(x, y), a(y)] + \beta U(x) \right\} f[x|y, a(y)]dx = U(y).$$

Our model (2.3) in Section 2 is the most restrictive version, (2.5) is less restrictive and (2.6) is the least restrictive. (2.3) guarantees the acceptance of the contract by the agent at any state, (2.5) guarantees the acceptance of the contract at the initial state, while (2.6) does not guarantee either. Consequently, the contract from (2.3) is time-consistent and thus needs no enforcement, but the contracts from (2.5) and (2.6) may need enforcement when the market condition changes. If (2.3) yields a solution, then (2.5) and (2.6) must yield solutions.

Among the three alternative setups, we consider only model (2.3) in the rest of the paper.

2. Analysis

In this section, we investigate a few properties of model (2.3).

Assumption 2. (Separable Utility Function). $u(s, a) = u(s) - c(a)$. 
By Assumption 2, assuming that the first-order approach (FOA) is valid, (2.3) becomes

\[
V(y) = \max_{a \in k, s \in S} \int \left\{ v[x - s(x, y)] + \alpha V(x) \right\} f(x|y, a) dx
\]

s.t. \[
\int \left\{ u[s(x, y)] + \beta U(x) \right\} f_a(x|y, a) dx = c'(a),
\]

\[
\int \left\{ u[s(x, y)] + \beta U(x) \right\} f(x|y, a) dx = c(a) + U(y).
\]

Let \( s = h(x, z) \) be the unique solution of \( \frac{v'(x-s)}{w(x)} = z \) for \( z > \frac{v'(x)}{w(0)} \). \( h \) is well defined and is strictly increasing in \( z \) (see the proof of Proposition 2). Let

\[
\phi(x, z) = \begin{cases} h(x, z), & \text{if } z > \frac{v'(x)}{w(0)}, \\ 0, & \text{otherwise}. \end{cases}
\]

**Proposition 2. (The Optimal Contract).** Under Assumption 2, assuming that the FOA is valid, the optimal contract is

\[
s(x, y) = \phi \left\{ x, \lambda(y) \frac{f_a}{f}[x|y, a(y)] + \mu(y) \right\},
\]

where \( \lambda(y) \) and \( \mu(y) \) are respectively the Lagrange multipliers of the IC and IR conditions in (3.1).

Proposition 2 suggests that given an immediate past performance \( y \), the solution formula for the contract is exactly the same as that for the static model. In the special case for which the principal is risk neutral, we have \( h(z) = (u')^{-1} \left( \frac{1}{z} \right) \) and

\[
\phi(z) = \begin{cases} h(z), & \text{if } z > \frac{1}{u'(0+)} \\ 0, & \text{otherwise}. \end{cases}
\]

Then,

\[
s(x, y) = \phi \left\{ \lambda(y) \frac{f_a}{f}[x|y, a(y)] + \mu(y) \right\}.
\]

For example, if \( u(z) = \frac{1}{1-\gamma} z^{1-\gamma} \), where \( \gamma \) is the RRA, we have \( h(z) = z^{\frac{1}{\gamma}} \) and \( u \circ h(z) = \frac{1}{1-\gamma} z^{\frac{1}{\gamma}-1} \) for \( z > 0 \). For this special \( u \) to have concave \( u \circ h \), we need \( \gamma \geq \frac{1}{2} \).

The monotonicity of \( s(x, y) \) in current performance \( x \), i.e., the positivity of \( \lambda(y) \), is guaranteed by the following proposition.

**Proposition 3. (Monotonicity).** Under Assumption 2, assuming that the FOA is valid and \( \frac{f_a}{f}(x|y, a) \) is increasing in \( x \), we must have \( \lambda(y) > 0 \) and \( \mu(y) \geq 0 \), for all \( y \in \mathbb{Y} \).
Consider a repeated version of the recursive model, for which the output process \( \{y_t\}_{t=0}^{\infty} \) follows \( y_{t+1} \sim f(\cdot|a_t) \), instead of \( y_{t+1} \sim f(\cdot|y_t, a_t) \), for a given density function \( f \). In this case, \( \bar{U}(y), V(y) \) and \( f(x|y, a) \) are independent of \( y \). Then, (2.3) becomes

\[
V = \max_{a \in A, \ a' \in A} \frac{1}{1 - \alpha} \int [v(x - s(x))f(x|a)]dx \quad (3.3)
\]

s.t. \( a \in \arg \max_{a, a' \in A} \int u[s(x), a']f(x|a')dx \), \( \frac{1}{1 - \beta} \int u[s(x), a]f(x|a)dx = \bar{U} \).

**Proposition 4. (First Best by Repetition).** If \( \alpha = \beta \), when \( \alpha \to 1 \), the solution of the repeated recursive model (3.3) converges to the first best with \( \bar{U} = 0 \).

3. The Recursive Model Under Risk Neutrality

3.1. A Semi-Linear Contract

Linear contracts have turned out to be very popular in applications of contract theory. The existence of a second-best linear contract was first found by Bhattacharyya–Lafontaine (1995) and was later extended by Kim–Wang (1998).

For the static model under risk neutrality, a linear contract exists, and it is not only simple and intuitive, but also the first best. Can we also have such a contract for a dynamic model?

**Proposition 5. (Semi-Linear Contract).** If both the principal and the agent are risk neutral, assuming that the first-order approach is valid, then there exists a semi-linear contract of the following form that induces the first-best effort \( a^*(y) \):

\[
s(x, y) = \varphi(y)x + \psi(y),
\]

where \( \varphi(y) \) and \( \psi(y) \) are defined by

\[
\varphi(y) = \frac{c'[a^*(y)] - \beta \int \bar{U}(x)f_a[x|y, a^*(y)]dx}{\int xf_a[x|y, a^*(y)]dx}, \quad (4.2a)
\]

\[
\psi(y) = \bar{U}(y) + c[a^*(y)] - \beta \int \bar{U}(x)f[x|y, a^*(y)]dx - \int xf[x|y, a^*(y)]dx \left\{ c'[a^*(y)] - \beta \int \bar{U}(x)f_a[x|y, a^*(y)]dx \right\}. \quad (4.2b)
\]
and where $V(\cdot)$ and $a^*(\cdot)$ are determined by

$$
\int \left[ x + \alpha V(x) + \beta \bar{U}(x) \right] f_a(x|y, a^*(y)) dx = c[a^*(y)], \quad (4.3a)
$$

$$
\int \left[ x + \alpha V(x) + \beta \bar{U}(x) \right] f[x|y, a^*(y)] dx = V(y) + \bar{U}(y) + c[a^*(y)]. \quad (4.3b)
$$

Given the solution in (4.1), under what conditions is the FOA valid? We find that if

$$
\int \bar{U}(x) f_a(x|y, a) dx \leq 0 \quad \text{and} \quad \int x f_a(x|y, a) dx \leq 0 \quad \text{for any } (a, y) \in A \times Y, \quad \text{and} \quad \varphi(y) \geq 0
$$

for any $y \in Y$, then the second-order condition (SOC) for the IC condition of $a^*(y)$ is satisfied for any $y \in Y$, i.e., the FOA is valid.

**Example 1** Let

$$
y_t = a_t + y_{t-1} + \varepsilon_t,
\varepsilon_t \sim N(0, \sigma^2),
\bar{U}(y) = by,
c(a) = \frac{1}{2} a^2,
$$

where $b \geq 0$ and $\sigma > 0$ are constants. Then,

$$
f(x|y, a) = \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{(x-a-y)^2}{2\sigma^2}}, \quad R(y, a) \equiv \int x f(x|y, a) dx = a + y,
$$

By Proposition 5, the solution is

$$
a^* = \frac{1 + b(\beta - \alpha)}{1 - \alpha},
$$

$$
s(x, y) = \frac{1 - \alpha b(1 - \beta)}{1 - \alpha} x - \frac{1 - b(1 - \beta)}{1 - \alpha} y - \frac{1}{2} \left[ \frac{1 + b(\beta - \alpha)}{1 - \alpha} \right]^2,
$$

$$
V(y) = \frac{1}{1 - \alpha} \left[ \frac{1 - (1 - \beta)b}{2(1 - \alpha)^3} y + \left[ 1 + b(\beta - \alpha) \right]^2 \right].
$$

The FOA is valid in this case. Due to discounting the future, the two parties share revenue under risk neutrality.

Many real-world contracts have simple sharing rules for profits, but such a sharing rule often evolves over time. For example, foreign direct investment (FDI) is typically carried out by two forms of contract: equity joint venture (EJV) and contractual joint venture (CJV). In an EJV, the two firms, a local firm and a foreign firm, agree upon a specific division of equity ownership and their future shares of profits equal their shares of equity. The shares of equity can change in the future if both sides agree to trade shares of equity between themselves, which leads to a corresponding change of profit sharing. On the other hand, in a CJV, profit
shares for all the contracting years are well specified, and equity sharing is irrelevant, and typically the equity is specified to belong to one of the parties by the end of the contract. In practice, EJVs tend to have a constant sharing rule over the contracting years, while CJVs tend to have a variable sharing rule. In fact, profit shares in CJVs may vary a lot from year to year.

There are many interesting observations on the evolution of sharing rule over time, but the research on it is very limited because of its complexity. One such an example is Tao–Wang (1998), who discuss the well-known puzzling phenomenon of increasing local firm’s share of revenue over the contracting years for all CJVs in China. In a special case of our semi-linear contract when the output grows at a constant rate $r$, we find that the agent’s share of revenue $s(x,y)$ is increasing in output, which is consistent with the observations in Tao–Wang (1998). In addition, this revenue share converges to a constant as output goes to infinity.

3.2. The Repeated Model Under Risk Neutrality

If we replace the density function $f(x|y,a)$ by $f(x|a)$, then we have a repeated model, which is a special case of Proposition 5. Let $R(a) \equiv \int x f(x|a)dx$ be the revenue function. In this case, $f(x|y,a)$, $\bar{U}(y)$ and $V(y)$ are independent of $y$, and thus Proposition 5 immediately gives the solution:

\[
s(x) = x + (1 - \beta)\bar{U} + c(a^*) - R(a^*),
\]
\[
V = \frac{R(a^*) - c(a^*) - (1 - \beta)\bar{U}}{1 - \alpha},
\]

where $a^*$ is determined by $R'(a^*) = c'(a^*)$. As expected, when $\alpha = \beta = 0$, we have the well-known first-best solution for the static model.

The Spear–Srivastava model is a repeated model, which can be compared with the repeated version of the recursive model. The Spear–Srivastava model under risk neutrality gives the following solution:

\[
s(x, w) = x + w - \beta\bar{u} + c(a^*) - R(a^*),
\]
\[
V(w) = \frac{R(a^*) - c(a^*)}{1 - \beta} - w,
\]

where $a^*$ is again determined by $R'(a^*) = c'(a^*)$, and $\bar{u} \equiv U(x, w)$ is a constant. Note that to avoid inconsistency, we must impose $\alpha = \beta$ for the Spear–Srivastava model under risk neutrality; also, $\bar{u}$ is undetermined and can be any arbitrary number.

Comparing the solution in (4.4) of the repeated recursive model with the solution in (4.5) of the Spear–Srivastava model, the two solutions are the same if $w = \bar{u} = \bar{U}$ and $\alpha = \beta$. Our model is more general as it allows $\alpha \neq \beta$.  

11
4. Computation

4.1. A Computing Algorithm

By the analysis in Section 3, solving model (2.3) is to find five functions \( s(x, y), a(y), V(y), \lambda(y), \mu(y) \), which are determined by the following five equations:

\[
V(y) = \int \left\{ v[x - s(x, y)] + \alpha V(x) \right\} f[x|y, a(y)] dx, \quad (5.1a)
\]

\[
\int \left\{ u[s(x, y)] + \beta \bar{U}(x) \right\} f_a[x|y, a(y)] dx = c'[a(y)], \quad (5.1b)
\]

\[
\int \left\{ u[s(x, y)] + \beta \bar{U}(x) \right\} f[x|y, a(y)] dx = \bar{U}(y) + c[a(y)], \quad (5.1c)
\]

\[
s(x, y) = \phi \left\{ x, \lambda(y) f_a[f(x|y, a(y)) + \mu(y)] \right\}, \quad (5.1d)
\]

\[
\lambda(y) \frac{\int \left\{ v[x - s(x, y)] + \alpha V(x) \right\} f_a[x|y, a(y)] dx}{c''[a(y)] - \int \left\{ u[s(x, y)] + \beta \bar{U}(x) \right\} f_a[a|x|y, a(y)] dx}, \quad (5.1e)
\]

for all \( y \in Y \). That is, the solution of (2.3) must satisfy these equations. Because there is generally no closed-form solution for (5.1), we now propose a computing algorithm that can be used to find a numerical solution. The computing procedure yields fast convergence of computation and it is easy to implement. For simplicity, assume that the principal is risk neutral; a risk-averse principal will only increase computational complexity slightly. We will also assume finite states; this assumption is needed for a numerical solution.

Assumption 3. (Risk Neutral Principal). The principal is risk neutral.

Assumption 4. (Finite States). \( Y \) is finite.

Let \( Y = (y_1, \ldots, y_n) \) and

\[
a_i \equiv a(y_i), \quad s_{ij} \equiv s(y_j, y_i), \quad V_i \equiv V(y_i), \quad \bar{U}_i \equiv \bar{U}(y_i), \quad p_{ij}(a_i) \equiv f(y_j|y_i, a_i),
\]

\[
\mu_i \equiv \mu(y_i), \quad \lambda_i \equiv \lambda(y_i).
\]

Then, the discrete time version of (5.1) is

\[
V_i = \sum_{j=1}^{n} (y_j - s_{ij} + \alpha V_j) p_{ij}(a_i),
\]

\[
\sum_{j=1}^{n} [u(s_{ij}) + \beta \bar{U}_j] p_{ij}'(a_i) = c'(a_i),
\]
\[
\sum_{j=1}^{n} [u(s_{ij}) + \beta U_j] p_{ij}(a_i) = \bar{U}_i + c(a_i),
\]

\[
s_{ik} = \phi \left[ \lambda_i - \frac{\mu_i}{p_{ij}(a_i)} \right], \quad k = 1, \ldots, n,
\]

\[
\lambda_i = \frac{\sum_j (y_j - s_{ij} + \alpha V_j) p'_{ij}(a_i)}{c''(a_i) - \sum_j [u(s_{ij}) + \beta U_j] p''_{ij}(a_i)},
\]

for \( i = 1, \ldots, n \). These five equations determine five variables \((a_i, s_{ij}, V_i, \lambda_i, \mu_i)\) for \( i, j = 1, \ldots, n \).

Starting with a given \( s \in \mathbb{R}^{n \times n} \), we compute in five simple steps based on the five equations; each step uses one equation.

1. Given \( s \), solve for \( a \) from

\[
\sum_j [u(s_{ij}) + \beta U_j] p'_{ij}(a_i) = c'(a_i), \quad i = 1, \ldots, n.
\]

Each \( a_i \) is determined independently by one equation in (5.2).

2. Given \( a \) and \( s \), calculate

\[
V = \begin{pmatrix}
1 - \alpha p_{11}(a_1) & -\alpha p_{12}(a_1) & \cdots & -\alpha p_{1n}(a_1) \\
-\alpha p_{21}(a_2) & 1 - \alpha p_{22}(a_2) & \cdots & -\alpha p_{2n}(a_2) \\
\vdots & \vdots & \ddots & \vdots \\
-\alpha p_{n1}(a_n) & -\alpha p_{n2}(a_n) & \cdots & 1 - \alpha p_{nn}(a_n)
\end{pmatrix}^{-1}
\begin{pmatrix}
\sum_j v(y_j - s_{1j}) p_{1j}(a_1) \\
\vdots \\
\sum_j v(y_j - s_{nj}) p_{nj}(a_n)
\end{pmatrix}.
\]

3. Given \( a, s \) and \( V \), calculate

\[
\lambda_i = \frac{\sum_j [v(y_j - s_{ij}) + \alpha V_j] p'_{ij}(a_i)}{c''(a_i) - \sum_j [u(s_{ij}) + \beta U_j] p''_{ij}(a_i)}, \quad i = 1, \ldots, n.
\]

4. Given \( a \) and \( \lambda \), solve for \( \mu \) from

\[
\sum_j u \circ \phi \left[ \lambda_i - \frac{\mu_i}{p_{ij}(a_i)} \right] p_{ij}(a_i) + \beta \sum_j p_{ij}(a_i) \bar{U}_j = \bar{U}_i + c(a_i), \quad i = 1, \ldots, n.
\]

Each \( \mu_i \) is determined independently by one equation in (5.5).

5. Finally, given \( a, \lambda \) and \( \mu \), calculate

\[
s_{ij} = \phi \left[ \lambda_i - \frac{\mu_i}{p_{ij}(a_i)} \right], \quad i, j = 1, \ldots, n.
\]

We can then go back to step 1 with the new \( s \) from step 5 for another round of calculation.
We can start from an arbitrary contract $s(0) \in \mathbb{R}^{n \times n}$, find a contract $s(1)$ from the five steps, then find $s(2)$ from the five steps, etc., until after $N$ such cycles the contract stabilizes, i.e., until $s(N)$ and $s(N+1)$ are practically equal. The limiting $s$ must satisfy the equations in (5.1). By this procedure, a computer can quickly find a solution at each step if it exists, and the sequence of solutions will typically converge very quickly. For the numerical example in next section, the five-step system typically converges in about 10 rounds.

4.2. A Numerical Example

We choose $A = \mathbb{R}_+$, $\mathcal{Y} = \{y_1, y_2\}$ with $y_1 < y_2$, and

$$f(y_1 | y, a) = e^{-ay}, \quad f(y_2 | y, a) = 1 - e^{-ay};$$

$U_1 = 10, \quad U_2 = 10.1;$

$\alpha = 0.98, \quad \beta = 0.98;$

$u(s) = \frac{1}{\gamma} (1 - e^{-\gamma s}), \quad \gamma = 3;$

$v(y) = y;$

$c(a) = \frac{1}{2} a^2.$

That is, there are only two possible states $y_1$ and $y_2$, the bad and good states, and the principal is risk neutral. These choices of function forms for a numerical solution is typical in contract theory. With this choice, increases in effort $a$ or past performance $y$ will increase the chance of getting the higher output $y_2$. We also have

$$\frac{f_a}{f}(y_1 | y, a) = -y, \quad \frac{f_a}{f}(y_2 | y, a) = \frac{ye^{-ay}}{1 - e^{-ay}},$$

which yields $\frac{f_a}{f}(y_2 | y, a) > \frac{f_a}{f}(y_1 | y, a)$. We also have $h(z) = \frac{1}{\gamma} \ln(z)$, for $z > 1$.

In the benchmark case, we choose $y_1 = 1$ and $y_2 = 2$, and the solution is

$$a = \begin{pmatrix} 0.15 \\ 0.17 \end{pmatrix}, \quad s = \begin{pmatrix} 0.273 & 0.533 \\ 0.610 & 0.786 \end{pmatrix}, \quad V = \begin{pmatrix} 39.92 \\ 39.68 \end{pmatrix}, \quad \lambda = \begin{pmatrix} 0.38 \\ 0.63 \end{pmatrix}, \quad \mu = \begin{pmatrix} 2.65 \\ 7.49 \end{pmatrix}.$$

Generally, we choose either $y_1 = 1$ or $y_1 = 1.5$, and a continuum of $y_2 : y_2 \in (1.5, \infty)$. The solutions for $a_i$ and $s_{ij}$ are depicted in the following six diagrams. The figures show those parts for which a solution exists.
We have a few observations about the diagrams. First, no matter whether the economy is in a bad or good state, the effort will always increase with an increase in the good-state output $y_2$, unless the limited liability condition in the case of ‘the good state followed by the bad state’ ($y_2 \rightarrow y_1$) forces the principal to reduce payment for the case of ‘the good state followed by the good state’ ($y_2 \rightarrow y_2$). There are two reasons for this result. A higher good-state output makes it worthwhile for the agent to work hard since the pay increase will compensate for his cost of a higher effort. Furthermore, an increased performance gap between the two states will also make it easier for the principal to relate output to effort.
Second, the contract payment to the agent is reduced when the output stays on the bad state for two consecutive periods. The reason may be that the two consecutive bad performances strengthen the principal’s belief that the agent is not working hard. But the reduction levels off when the good state is really good. A sharp difference in outputs for the two states leads the principal to believe that even two consecutive bad performances can still be due to a bad luck rather than low effort, since a small increase in effort with negligible cost to the agent could lead to a large performance improvement and a large reward — the agent should have realized this and should have chosen to work hard if the state were good. Symmetrically, the contract payment to the agent increases in the case of $y_2 \rightarrow y_2$ unless the limited liability condition limits the punishment to the agent in the case of $y_2 \rightarrow y_1$.

Third, the contract payment to the agent increases substantially if the output moves from being bad to good, and conversely, the contract payment to the agent drops dramatically if the output moves from being good to bad. The better the good state is, the larger the reward and punishment are. When limited liability limits the punishment on the agent, the principal reduces payment in the case of $y_2 \rightarrow y_2$ to compensate for the case of $y_2 \rightarrow y_1$.

Fourth, an increase in the bad-state output $y_1$ will increase the effort in the bad state but will reduce the effort in the good state. Since an increase in the bad-state output narrows the gap between the performances in the two states, the agent distributes his effort more evenly between the two states. In essence, there is a substitution of efforts between the two states; efforts will be more evenly distributed if outputs are more evenly distributed.

Finally, in contrast with the fourth point, when the output in the bad state is increased, the contract payment will be reduced for the cases of $y_1 \rightarrow y_1$, $y_2 \rightarrow y_2$ and $y_1 \rightarrow y_2$, but the contract payment will be increased for the case of $y_2 \rightarrow y_1$. This can be explained by the fact that an improvement of performance in the bad state reduces the gap of performances between the two states, which leads to a more evenly distributed contract pay scheme. Notice that the increase in payment in the case of $y_2 \rightarrow y_1$ is marginal even if the improvement in the bad-state output is substantial.

5. Concluding Remarks

This paper proposes a dynamic agency model based on a Markov state process. It is a generalization of the existing repeated agent model. We derive the solution equations and discuss some characteristics of the solution. Two special cases for which the first best can be achieved are discussed. The semi-linear contract is of particular interest. By the popularity of the linear contract for the static model, we expect many applications for the semi-linear dynamic contract. However, a closed-form solution is unavailable for the general model, which is also true for the static basic agency model. We thus propose a computing algorithm using the well-known software called Mathcad. Mathcad is a powerful yet very user friendly software program in Windows 95. A numerical example is done using the computing algorithm, from which we can observe a few characteristics of a dynamic contract.
There are many hard technical problems left to be done. First, we need an existence result for the solution of equation set (5.1). This amounts to finding a domain of output \(y\) within which a solution of (5.1) is guaranteed. Second, we only know that any solution of problem (2.3) must be a solution of equation set (5.1); we need the uniqueness of solution for (5.1) to guarantee that any solution of (5.1) is in fact a solution of (2.3). Since problem (2.3) itself has already imposed quite stringent conditions, we expect equation set (5.1) to have a unique solution under mild conditions. Third, for the computing algorithm, the difficult question is the convergence of the iterative process. This also amounts to finding a domain of output \(y\) within which a solution of the five-step algorithm converges. Finally, in a dynamic model, since past performances may reveal a lot of information to the principal, as our numerical example clearly shows, there may exist many other types of first-best contracts, other than the one for the repeated model.

References


