An Efficient Bonus Contract

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Abstract: Why are bonus/promotion schemes so widely used in reality? Are they effective in alleviating incentive problems? For the standard agency model, this paper proposes an alternative solution to the classical solution in Holmström (1979). The advantages of our solution are that (1) it is a simple solution; (2) it is not based on the troublesome first-order approach; and (3) it is the first best. The disadvantage of our solution is that it imposes a boundary condition on the support of the distribution function. However, this boundary condition does have sensible economic interpretations and our optimal contract resembles the widely observed bonus/promotion scheme in reality. Hence, our solution shows the potential of a bonus/promotion scheme in resolving incentive problems.

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1. Introduction

The objective of this paper is to explain why bonus/promotion schemes are so popular in reality. We do this by finding an alternative optimal contract for the standard agency model. As we known, when there is no risk aversion, there is a simple optimal contract: a linear contract. Linear contracts are popular in reality. However, when there is risk aversion, the optimal contract becomes very complex, which severely prevents its use in applications. This paper finds two simple conditions under which the standard agency model with risk aversion has a simple optimal contract: a bonus contract. This contract is convenient for applications and it is implementable by a simple bonus/promotion scheme.

Real-world contracts between principals and agents are typically very simple and often have a multi-step bonus structure that specifies wage increases for certain minimum levels of performance. Why can contracts be so simple in reality? This paper attempts to provide an answer by establishing the optimality of a one-step contract for the standard agency model. Our optimal contract is equivalent to a bonus scheme by which an agent is rewarded a bonus upon achieving a minimum performance. Such a bonus, reward or promotion scheme is widely observed in reality and is generally believed to be a key incentive device.

The classical solution to the standard principal-agent model is based on the first-order approach (FOA). The foundation of this solution is laid down by Mirrlees (1999) and Holmström (1979). However, due to its complex structure, the existence of the solution (Mirrlees 1999 and Holmström 1979), the validity of the FOA (Rogerson 1985 and Jewitt 1988) and the monotonicity of the solution (Grossman–Hart 1983) are all difficult issues. As a result, the classical solution remains elusive to those applied fields where risk aversion is a necessity. In contrast, our bonus contract does not face these difficulties. In fact, we find simple conditions under which (1) the solution is globally optimal and (2) the incentive-compatibility (IC) condition is satisfied either locally or globally.

This paper proceeds as follows. We present the agency model in the next section. The solution is derived in Section 3. Discussion about our solution is made in Section 4. Two examples are presented in Section 5. Finally, we conclude the paper with some remarks in Section 6.

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2 Bhattacharyya–Lafontaine (1995) are the first to provide such a result for an additive output process of the form \( x = a + \varepsilon \). Kim–Wang (1998) provide a general result.
2. The Model

Let \( a \) be the agent’s effort and \( A \) be the effort space. Let the contract space be

\[
S \equiv \{ s : \mathbb{R} \to \mathbb{R}_+ | s \text{ is Lesbegue integrable} \},
\]

where we have implicitly imposed the limited liability condition: \( s(x) \geq 0 \), just as Mirrlees (1999) and Holmström (1979) did. Let the density function of output be \( f(x,a) \) for \( A(a) \leq x \leq B(a) \) and zero elsewhere, where the domain is \( D(f) \equiv [A(a), B(a)] \). We allow \( B(a) \) to be arbitrary, including being a constant or \( B(a) = +\infty \). But, we will impose a condition on \( A(a) \) later. Given a utility function \( u : \mathbb{R}_+ \to \mathbb{R}_+ \), the agent’s expected utility is

\[
U(a,s) = \int_{A(a)}^{B(a)} u(s(x)) f(x,a) dx - c(a).
\]

Given a contract \( s(\cdot) \), the agent’s problem is

\[
\max_{a \in \mathcal{A}} \int_{A(a)}^{B(a)} u(s(x)) f(x,a) dx - c(a).
\]

Thus, the principal’s problem is

\[
\pi \equiv \max_{s \in \mathcal{S}, a \in \mathcal{A}} \int_{A(a)}^{B(a)} [x - s(x)] f(x,a) dx
\]

s.t. \( IR : \int_{A(a)}^{B(a)} u(s(x)) f(x,a) dx \geq c(a) \), \( IC : a \in \arg\max_a U(a',s) \).

Here, the reservation value is taken to be \( \bar{u} = 0 \). If \( \bar{u} \neq 0 \), we can replace \( c(a) \) by \( C(a) \equiv c(a) + \bar{u} \) and the model remains the same.

This model is the same as the standard agency model formulated in Holmström (1979) except that we allow the boundaries of the domain to be dependent on effort. It turns out that this dependence is important for our alternative solution to the classical solution in Holmström (1979). Another difference is that we do not rely on the FOA.

**Assumption 1.** The utility function \( u(x) \) is concave, onto and strictly increasing.

**Assumption 2.** The expected revenue \( R(a) \equiv \int_{A(a)}^{B(a)} x f(x,a) dx \) is increasing and concave.

**Assumption 3 (FOSD).** \( F_a(x,a) \leq 0 \) for any \( x \in \mathbb{R} \) and \( a \in \mathcal{A} \).
These three assumptions are natural requirements. In particular, \( u \) being onto and strictly increasing is to ensure \( u^{-1}[c(a)] \) to be well defined, and Assumption 3 is necessary for any sensible contract theory.

3. The Solution

We now derive the solution of model (1). Given optimal contract \( s^*(x) \) together with optimal effort \( a^* \), as shown in Section 3.3 later, \( U(a,s^*) \) is generally not continuously differentiable at \( a = a^* \). Hence, the first-order condition \( U_s(a^*,s^*) = 0 \) cannot replace the IC condition in problem (1). This means that the FOA is no longer applicable to our model. We need to use a new approach for this model. Our approach is to consider the first-best version of problem (1) by ignoring the IC condition first and later to show that the first-best solution is also the solution of problem (1).

**Proposition 1.** Let \( a^* \) be the solution of the following equation:

\[
\frac{\partial u^{-1}[c(a^*)]}{\partial a} = R'(a^*),
\]

and suppose that the following two conditions are satisfied:

\[
\frac{\partial \ln u^{-1}[c(a^*)]}{\partial a} \geq \frac{\partial \ln c(a^*)}{\partial a},
\]

\[
-F_s[A(a^*),a^*] > \frac{\partial \ln c(a^*)}{\partial a}.
\]

Then, under Assumptions 1–3, the optimal effort is \( a^* \) and the optimal contract is

\[
s^*(x) = \begin{cases} 
0 & \text{if } x < A(a^*), \\
u^{-1}[c(a^*)] & \text{if } x \geq A(a^*).
\end{cases}
\]

Furthermore, \( (a^*,s^*) \) achieves the first best.

**Proof.** We proceed the proof by three steps.
Step 1. The Optimal Contract

Without the IC condition, the Lagrangian for problem (1) is

\[
L = \int_{A(a)}^{B(a)} [x - s(x)]f(x, a)dx + \lambda \int_{A(a)}^{B(a)} u[s(x)]f(x, a)dx - c(a),
\]

where \( \lambda \geq 0 \) is a Lagrange multiplier. The Hamiltonian is

\[
H(x, s) = x - s + \lambda u(s).
\]

The Euler equation \( H_x = 0 \) implies

\[
\lambda u'[s(x)] = 1. \tag{6}
\]

Thus, by Pontryagin’s maximum principle, the optimal contract is

\[
s^*(x) = \begin{cases} 
0 & \text{if } x < A(a^*), \\
\frac{1}{\lambda} & \text{if } x \geq A(a^*),
\end{cases}
\]

where \( u'(\bar{x}) \equiv 1/\lambda \). Note that a jump at \( A(a^*) \) is necessary for satisfying the IC condition, as shown later in Section 3.3. By the IR condition in (1), we have

\[
c(a^*) = \int_{A(a^*)}^{B(a)} u(\bar{x})f(x, a^*)dx = u(\bar{x}),
\]

implying

\[
\bar{x} = u^{-1}[c(a^*)]. \tag{7}
\]

Here, \( u(x) \) being onto and strictly increasing ensures the existence and uniqueness of \( \bar{x} \). Thus,

\[
s^*(x) = \begin{cases} 
0 & \text{if } x < A(a^*), \\
u^{-1}[c(a^*)] & \text{if } x \geq A(a^*). \tag{8}
\end{cases}
\]

Note that this optimal contract is dependent on the optimal effort \( a^* \), which is determined in the next step.

Step 2. The Optimal Effort

Given \( s^*(x) \) in (8), the Lagrangian becomes

\[
L(a) = \int_{A(a)}^{B(a)} x f(x, a)dx - \int_{A(a)}^{B(a)} s^*(x)f(x, a)dx + \lambda \int_{A(a)}^{B(a)} u[s^*(x)]f(x, a)dx - c(a).
\]

If \( a < a^* \),
Thus,

\[
L'(a) = R'(a) - \lambda c'(a) + F_x[A(a^*), a]\{u^{-1}[c(a^*)] - \lambda c(a^*)\}, \quad \text{for } a < a^*. \tag{9}
\]

If \(a \geq a^*\),

\[
L(a) = R(a) - \overline{\alpha} \int_{\alpha(a^*)}^{B(a^*)} f(x, a) \, dx + \lambda u(\overline{\alpha}) \int_{\alpha(a^*)}^{B(a^*)} f(x, a) \, dx - c(a)
\]

\[
= R(a) - \overline{\alpha} + \lambda [u(\overline{\alpha}) - c(a)].
\]

Thus,

\[
L'(a) = R'(a) - \lambda c'(a), \quad \text{for } a > a^*. \tag{10}
\]

To ensure the optimality of \(a^*\), let

\[
R'(a^*) - \lambda c'(a^*) = 0. \tag{11}
\]

Since \(R(a) - \lambda c(a)\) is concave in \(a\) or \(R'(a) - \lambda c'(a)\) is decreasing in \(a\), (11) implies \(L'(a) \leq 0\) for \(a \geq a^*\). Suppose

\[
u^{-1}[c(a^*)] - \lambda c(a^*) \leq 0. \tag{12}
\]

Then, by Assumption 3 and the decreasingness of \(R'(a) - \lambda c'(a)\), (11) implies \(L'(a) \geq 0\) for \(a \leq a^*\). In other words, conditions (11) and (12) imply the global optimality of \(a^*\).

Notice that \(L(a)\) is indeed continuous at \(a^*\). By (6) and (7),

\[
\frac{1}{\lambda} = u'[u^{-1}[c(a^*)]].
\]

By (11), this becomes

\[
\frac{c'(a^*)}{R'(a^*)} = u'[u^{-1}[c(a^*)]]. \tag{13}
\]

Since

\[
\frac{\partial u^{-1}[c(a)]}{\partial a} = \frac{c'(a)}{u'[u^{-1}[c(a)]]},
\]

(13) becomes

\[
\frac{\partial u^{-1}[c(a^*)]}{\partial a} = R'(a^*). \tag{14}
\]
This equation determines $a'$. Also, condition (12) can be written as

$$\frac{c(a')}{u^{-1}(c(a'))} \geq u'[u^{-1}(c(a'))],$$

implying

$$\frac{\partial \ln u^{-1}(c(a'))}{\partial a} \geq \frac{\partial \ln c(a')}{\partial a}. \tag{15}$$

In summary, we need equations (14) and (15), where equation (14) determines $a'$ and condition (15) ensures the global optimality of $a'$.

**Step 3. The IC Condition**

We now verify the IC condition in (1). For the optimal contract in (8), if $a < a^*$,

$$U(a, s^*) = \int_{\lambda(a)}^{\beta(a)} u(x) f(x, a) dx - c(a) = c(a') \int_{\lambda(a')}^{\beta(a')} f(x, a) dx - c(a)$$

$$= c(a') \left[ 1 - F(A(a'), a) \right] - c(a),$$

implying

$$U_a(a, s^*) = -c(a') F_a[A(a'), a] - c'(a), \quad \text{for } a < a^*,$$

implying

$$U_a(a^*, s^*) = -c(a') F_a[A(a^*), a^*] - c'(a^*).$$

If $a \geq a^*$,

$$U(a, s^*) = \int_{\lambda(a)}^{\beta(a)} u(x) f(x, a) dx - c(a) = c(a') \int_{\lambda(a')}^{\beta(a')} f(x, a) dx - c(a) = c(a') - c(a).$$

We have $U(a^*, s^*) = 0$ and $U(a, s^*) < U(a^*, s^*)$ when $a > a^*$. Thus, for $a^*$ to be locally optimal, we need to find a $\delta > 0$ such that $U(a, s^*) < 0$ when $a \in (a^* - \delta, a^*)$. This is guaranteed by $U_a(a^*, s^*) > 0$ or

$$-F_a[A(a^*), a^*] > \frac{c'(a^*)}{c(a^*)}. \tag{16}$$

Notice that $U(a, s^*)$ is indeed continuous at $a^*$. In other words, under condition (16), given contract $s^*(x)$, $a^*$ is indeed a local maximum point of $U(a, s^*)$. That is, the IC condition is satisfied.

The required conditions (14), (15) and (16) are listed in the proposition. The proof is now complete. Q.E.D.
4. Discussion

We have now shown that a bonus contract is optimal for the standard agency model with risk aversion. As the bonus is \( u^{-1}[c(a)] \), condition (2) means that the marginal bonus is equal to the marginal revenue. Condition (3) means that the marginal elasticity of bonus is larger than the marginal elasticity of cost. And, condition (4) means that the marginal chance of getting the bonus is higher than the marginal elasticity of cost.

Although the contract is derived as a revenue-sharing contract, it is implementable by a simple a bonus or promotion scheme. A labor contract in reality often has such a component. In reality, a contract may contain two reward schedules \( s_b(x) \) and \( s_m(x) \), where \( s_b(x) \) is a bonus scheme that may be firm-specific and \( s_m(x) \) is the market wage that depends on marketable skills. We have only derived the firm-specific bonus scheme in this paper. The market wage is a separate issue, which will relate to market conditions.

Our solution looks strikingly different from the FOA-based classical solution in Holmström (1979). Our solution looks puzzling at first glance because of its simplicity. However, the intuition for the solution is actually fairly simple. To induce the optimal effort \( a^* \), the FOA suggests that the principal should associate each output level to a payment. Alternatively, our bonus approach suggests that the principal can induce \( a^* \) by simply offering a bonus at a minimum level of performance. For the latter strategy to work, the dependence of the distribution function on effort at the minimum level of performance is crucial. Condition (4) is precisely for such dependence.

There are some interesting features in our solution. First, it is a closed-form solution with a clear bonus component. In fact, the bonus is paid at a minimum level of performance. This bonus can be in the form of a promotion, a bonus, a reward, or a change of nature of the contract. Second, the solution has an interesting form with simple and intuitive expressions determining the pay and effort. Third, the solution achieves the first best.

We have some further technical remarks about the conditions in the proposition. First, since both \( c(\cdot) \) and \( u^{-1}(\cdot) \) are convex, \( u^{-1}[c(a)] \) is convex in \( a \), implying that
\[
\frac{\partial u^{-1}[c(a)]}{\partial a} \text{ is increasing in } a. \text{ Also, since } R'(a) \text{ is decreasing in } a, \text{ the optimal effort } a^* \text{ from (2) must be unique.}
\]

Second, since \( u^{-1} \) is convex, \( u^{-1}[c(a)] \) is more convex than \( c(a) \). This indicates that condition (3) can be easily satisfied for any \( a \in \mathbb{A} \). For example, for \( u(x) = x^{1-\alpha} \) and \( c(a) = a^\beta \), where \( \alpha \in [0, 1] \) is the relative risk aversion and \( \beta \geq 1 \), (3) is satisfied for any \( a \in \mathbb{R}_+ \).

Third, there is no requirement on the right boundary \( B(a) \) at all.

Fourth, by the same derivation as in the proof, \( a^* \) is the global maximum point of \( U(a, s^*) \) if (16) is replaced by the following stronger version:

\[
-F_u[A(a^*), a] \geq \frac{c'(a)}{c(a^*)}, \text{ for any } a < a^*.
\]

Finally, we have

\[
F_u[A(a), a] = -f[A(a), a]A'(a), \text{ for any } a \in \mathbb{A}.
\]

Equation (2) indicates that \( a^* \) is independent of many features in the distribution function. For example, if \( x = A(a) + \varepsilon \), where \( \varepsilon \geq 0 \) is a random variable with mean \( \bar{\varepsilon} \), and if \( F_\varepsilon(x) \) and \( f_\varepsilon(x) \) are respectively the distribution and density functions of \( \varepsilon \), then \( F(x, a) = F_\varepsilon[x - A(a)] \), implying

\[
R(a) = A(a) + \bar{\varepsilon}, \quad -F_u[A(a), a] = f_\varepsilon(0)A'(a).
\]

Then, equation (2) and condition (4) respectively become

\[
\frac{\partial u^{-1}[c(a^*)]}{\partial a} = A'(a^*), \quad (19)
\]

\[
f_\varepsilon(0) > \frac{1}{A'(a^*)} \frac{\partial \ln c(a^*)}{\partial a}. \quad (20)
\]

From (19), we can see that \( a^* \) has nothing to do with the distribution function \( F_\varepsilon(x) \). This means that the right-hand side of (20) is independent of the distribution function. Hence, condition (20) defines the class of density functions \( f_\varepsilon(x) \) on \([0, \infty)\) that has a lower bound at the left boundary. For example, if \( \varepsilon \) follows the exponential distribution with \( f_\varepsilon(x) = \frac{1}{\sigma} e^{-\frac{x}{\sigma}} \) for \( x \geq 0 \), where \( \sigma > 0 \) is the standard variance, then we have \( f_\varepsilon(0) = \frac{1}{\sigma} \), which satisfies (20) when \( \sigma \) is sufficiently small.
5. Examples

We consider two examples, which use the exponential distribution and the uniform distribution, respectively.

5.1. Example 1

Suppose

\[ u(x) = x^{1-\alpha}, \]
\[ c(a) = a^\beta, \]
\[ f(x, a) = \frac{1}{a} e^{-\frac{x - A(a)}{a}}, \text{ for } x \geq A(a), \]

where \( A(a) \) is an arbitrary increasing function, \( \alpha \in [0, 1] \) is the relative risk aversion, and \( \beta \geq 1 \). This set of functions includes the example in Holmström (1979) as a special case. We have

\[ u^{-1}[c(a)] = a^{1-\alpha}, \quad R(a) = A(a) + a, \quad F_a[A(a), a] = -\frac{A'(a)}{a}. \]

Condition (2) becomes

\[ \alpha \beta a^\beta (a^*)^{\beta-1} = 1 + A'(a^*). \quad (21) \]

Condition (3) is satisfied for any \( a \geq 0 \). Condition (4) becomes

\[ \beta < A'(a^*). \quad (22) \]

Let \( A(a) = \gamma a \) for some \( \gamma > 1 \). Then, (22) is satisfied for any \( a \in \mathbb{A} \) and for \( \beta \leq \gamma \); in fact, the stronger version of (4), condition (17), is also satisfied. By (21), we find

\[ a^* = \left[ \frac{(1 + \gamma)(1 - \alpha)}{\beta} \right]^{1-\alpha}, \quad \pi = \left[ \frac{(1 + \gamma)(1 - \alpha)}{\beta} \right]^{\beta}. \]
5.2. Example 2

Suppose

\[ u(x) = x^{1-\alpha}, \]
\[ c(a) = a^\beta, \]
\[ f(x, a) = \frac{1}{\sigma}, \quad \text{for } A(a) \leq x \leq A(a) + \sigma, \]

where \( A(a) \) is an arbitrary increasing function, \( \alpha \in [0, 1] \) is the relative risk aversion, \( \sigma > 0, \) and \( \beta \geq 1. \) We have

\[ u^{-1}[c(a)] = a^{\frac{\beta}{1-\alpha}}, \quad R(a) = A(a) + \frac{\sigma}{2}, \quad F_s[A(a), a] = -\frac{1}{\sigma} A'(a). \]

Condition (2) becomes

\[ \frac{\beta}{1-\alpha} (a^*)^{-\frac{\beta}{1-\alpha} - 1} = A'(a^*). \quad (23) \]

Condition (3) is satisfied for any \( a \geq 0. \) Condition (4) becomes

\[ \sigma < \frac{1}{\beta} a^* A'(a^*), \quad (24) \]

which is satisfied when \( \sigma \) is sufficiently small. Again, let \( A(a) = \gamma a \) for some \( \gamma > 1. \) Then, the stronger version of (4), condition (17), is also satisfied. By (23), we find

\[ a^* = \left[ \frac{\gamma(1-\alpha)}{\beta} \right]^{\frac{1-\alpha}{\alpha+\beta-1}}, \quad \overline{a} = \left[ \frac{\gamma(1-\alpha)}{\beta} \right]^{\frac{\beta}{\alpha+\beta-1}}. \]

6. Concluding Remarks

For the standard agency model, this paper proposes an alternative solution to the classical solution in Holmström (1979). There are several advantages of our solution. First, it is a simple closed-form solution and the terms of the solution have straightforward economic interpretations. Second, it is not based on the FOA. This avoids the restrictions implied by the validity of the FOA. Given a set of specific functions, the restrictions from the FOA typically have complex and unclear implications. Finally, it is the first best. The disadvantage of our solution is that it imposes a restriction on the distribution function, although we argue that the
requirement of a minimum level of performance for a bonus is consistent with many realistic situations.³

The classical solution can be a simple contract, a linear contract, only in a very special case when both the agent and the principal are risk neutral. However, it is shown by Kim–Wang (1998, 2004) that, if there is tiny risk aversion, the optimal contract is completely different from this linear contract. In contrast, our bonus contract is robust in the sense that when risk aversion diminishes, its limit is still a bonus contract.

Interestingly, our solution may offer a different view on the recent debate on manager’s pay. Economists have generally assumed that to induce the right incentives, a manager's pay must be closely associated with performance. This view is consistent with a FOA-based solution. However, our solution seems to be in favor of the old way (or the German-Japanese way) in which a bonus is associated with some minimum performance and a close association of pay to performance is unnecessary.

Our solution looks very much like a typical employment contract, in which a fixed wage rate is given based on a minimum level of education and working experience. We can interpret the boundary term $A(a)$ as the requirement for firm-specific skills. A product may be produced by many different technologies/techniques. The best or the most suitable technology for a firm's product tends to produce the product with the highest quality or the most desirable features and thus yields the highest value in the market. In other words, the value of those units of the product produced by the best technology will surpass a threshold of value dependent on a particular technology. Hence, firm-specific skills can be defined by a minimum output value. To surpass this value, an employee needs some firm-specific skills. To acquire these skills, an employee may need some talent and working experience and may need to invest in education and training programs. Thus, by this viewpoint, in the agency model, the lower boundary of the domain of the distribution function of output should generally be dependent on the agent’s effort. The classical solution in Holmström (1979) cannot allow such dependence.

Finally, our solution may explain franchising, by which a local franchisee pays the franchiser a fee for its brand name, where a brand name is typically associated with a minimum level of quality and market acceptance.

³ Discussions with similar approaches existed in as early as 1970s. See, for example, Gjesdal (1976) and Gravelle–Rees (1992, p.702). However, to our knowledge, no publication exists for a formal treatment of the standard agency model using a similar approach as ours. This is surprising considering the usefulness of a simple contractual solution.
References


